SEMINAR ON BEZRUKAVNIKOV'S EQUIVALENCE

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Abstract. These are the notes for RTG learning seminar on Bezrukavnikov's equivalence at University of Michigan in 2024 Fall. The notes taker is the only one who is responsible for all the mistakes and typos.

Contents

1. Introduction to Hecke algebra - Elad Zelingher

1.1. **Bi-invariant K-Hecke algebra of G.** Let F be a non-archimedean local field with ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{p}_F$, uniformizer ϖ_F, residual field $\mathbb{F}_q$. Let G be a split connected reductive group. Fix a Borel subgroup $B = AN$.

Definition. The *co-characters* of A is

$$X_*(A) = \mathrm{Hom}(\mathbb{G}_m, A).$$

Fix $\kappa = G(\mathcal{O}_F)$ to be a maximal compact open subgroup. Then $A(\mathcal{O}_F) = A \cap \kappa$. Let $A(\varpi) = A/A(\mathcal{O}_F)$.

Example 1.1. Let $G = \mathrm{GL}_n$ and A be the diagonal torus. Then we have identification

$$A(\varpi) = \left\{ \mathrm{diag}(\varpi^{k_1}, \cdots, \varpi^{k_n}) \mid k_1, \cdots, k_n \in \mathbb{Z} \right\}.$$

We have an isomorphism $X_*(A) \cong A(\varpi)$ given by $\check{\mu} \mapsto \check{\mu}(\varpi)$.

We will talk about the algebra $\mathcal{H}(G /\!\!/ K) = \mathcal{H}(K\backslash G/K)$for some compact open subgroup $K \subseteq G(F)$, where

$$\mathcal{H}(G /\!\!/ K) = \{f \in C_c^\infty(G) \mid f(k_1 g k_2) = f(g), \forall k_1, k_2 \in K, g \in G\}.$$

Today we will be interested in the cases

- $K = G(\mathcal{O})$;
- $K =$ the Iwahori subgroup.

Why study this? We want to study representations π of $G(F)$ that have a fixed K-vector. Every such π admits a representation of $\mathcal{H}(G /\!\!/ K)$.

1.2. **Spherical Hecke algebra.** This is the case $K = G(\mathcal{O}_F)$. $\mathcal{H}(G /\!\!/ F)$ is an algebra, the product being convolution

$$(f_1 * f_2)(g) = \int_{G(F)} f_1(x^{-1}) f_2(xg) dx.$$

For $K = G(\mathcal{O}_F)$, we have the Cartan decomposition

$$G(F) = \sqcup_{\check{\mu} \in X_*(A)_{\mathrm{dom}}} G(\mathcal{O}_F) \varpi_F^{\check{\mu}} G(\mathcal{O}_F).$$

Here, $X_*(A)_{\mathrm{dom}}$ consists of *dominant* co-characters. By applying the Gelfand's trick to involution, we have that

Proposition 1.1. *$\mathcal{H}(G /\!\!/ K)$ is commutative if $K = G(\mathcal{O}_F)$.*

Example 1.2. For $G = \mathrm{GL}_n$, the involution $g \mapsto {}^t g$ fixes the diagonal matrices and sends $\mathrm{GL}(\mathcal{O}_F)$ to itself.

Remark. In general, $\mathcal{H}(G /\!\!/ K)$ is not commutative. However, if $\check{\mu}_1, \check{\mu}_2$ are dominant characters, then we have

$$\chi_{K \varpi^{\check{\mu}_1} \varpi^{\check{\mu}_2} K} = \chi_{K \varpi^{\check{\mu}_1} K} * \chi_{K \varpi^{\check{\mu}_2} K}.$$

Therefore, R_K^+, the subalgebra of $\mathcal{H}(G /\!\!/ K)$ generated by $\chi_{K \varpi^{\check{\mu}} K}$, then R_K^+ is commutative and we have

$$\mathcal{H}(G /\!\!/ K) = \mathcal{H}(\kappa /\!\!/ K) * R_K^+ * \mathcal{H}(\kappa /\!\!/ K).$$

Here $\mathcal{H}(\kappa /\!\!/ K)$ is non-commutative but finite dimensional, and R_K^+ is commutative but infinite dimensional.

1.3. **Iwahori-Matsumoto Hecke algebra.** Consider the quotient map $G(\mathcal{O}) \twoheadrightarrow G(\mathbb{F}_q)$. Let I be the pre-image of $B(\mathbb{F}_q)$ under this quotient map.

Let W be the Weyl group of G and $\tilde{W} = W \ltimes A(\varpi)$ be the extended affine Weyl group. We have the *Bruhat-Iwahori decomposition*:

$$G(F) = \bigsqcup_{w \in \tilde{W}} IwI.$$

For $w \in \tilde{W}$, $f_w = \chi_{IwI}$ form a basis of $\mathcal{H}(G(F) /\!\!/ I)$. The algebra $\mathcal{H}(G(F) /\!\!/ I)$ is isomorphic to the *Iwahori-Matsumoto algebra.* The latter is defined as follows. $\tilde{W}$ has a decomposition

$$\tilde{W} = W_{\mathrm{aff}} \rtimes \pi_1(G).$$

W_{aff} is a Coxeter group, so it is equipped with a standard presentation with generators and relations. The Iwahori-Matsumoto algebra is the Hecke algebra associated to the Coxeter system (i.e. for each simple reflection s there is a T_s, and for each generator h of $\pi_1(G)$ there is T_h, satisfying all the relations of the Coxeter system and semi-direct product except that $s^2 = 1$ is replaced by $T_s^2 = qT_s + q$). The isomorphism is given by $f_w = h \ltimes (\prod_j s_{i_j}) \mapsto T_h \cdot T_{s_{i_1}} \cdot \dots \cdot T_{s_{i_r}}$.

Let $R_I^+ = \{f_{\varpi^{\check{\mu}}} \mid \check{\mu} \text{ is dominant}\}$. One can show f_w is always invertible. Define

$$R_I = \langle f_{\varpi^{\check{\mu}}}, f_{\varpi^{\check{\mu}}}^{-1} \mid \check{\mu} \text{ is dominant} \rangle.$$

This is a commutative subalgebra of $\mathcal{H}(G /\!\!/ I)$. Tempting to define a map

$$\begin{aligned} A(\varpi) &\to R_I^\times \\ \varpi^\mu &\mapsto f_{\varpi^\mu}. \end{aligned}$$

But this will not be a homomorphism of groups. If $\check{\mu}_1, \check{\mu}_2$ are dominant, then

$$f_{\varpi^{\mu_1} \varpi^{\mu_2}} = f_{\varpi^{\mu_1}} * f_{\varpi^{\mu_2}}.$$

Any co-character $\check{\lambda}$ is of the form $\check{\lambda} = \check{\mu}_1 (\check{\mu}_2)^{-1}$ where $\check{\mu}_1$ and $\check{\mu}_2$ are dominant. For $w \in \tilde{W}$, write

$$T_w = q^{-\frac{l(w)}{2}} f_w$$

where $l : \tilde{W} \to W_{\text{aff}} \to \mathbb{Z}_{\geq 0}$ is the length. Define for $\check{\lambda}$

$$\theta(\check{\lambda}) = (T_{\check{\mu}_1}) \cdot (T_{\check{\mu}_2})^{-1}.$$

Then θ is well-defined and is a homomorphism of groups

$$\theta : A(\varpi) \to R_I^{\times}.$$

1.4. **Bernstein-Zelevinsky relation.**

Theorem 1.1. *Let s be a simple reflection in W. We have the following identity*

$$\theta(\check{\lambda})T_s - T_s\theta(s(\check{\lambda})) = T_s\theta(\check{\lambda}) - \theta(s(\check{\lambda}))T_s = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\frac{\theta(\check{\lambda}) - \theta(s(\check{\lambda}))}{1 - \theta(\check{\alpha})^{-1}}.$$

where $\check{\alpha}$ is a certain co-character $s(\check{\alpha}) = \check{\alpha}^{-1}$.

Corollary 1.1. *The center of $\mathcal{H}(G /\!\!/ I)$ is*

$$\left\{ \sum_{\check{\lambda}} a_{\check{\lambda}}\theta(\check{\lambda}) \mid a_{w\check{\lambda}} = a_{\check{\lambda}}, \forall w \in W, \check{\lambda} \in X_*(A)_{\text{dom}} \right\}.$$

2. The nilpotent cone, Springer fibers & resolution and Steinberg variety - Alexander Hazeltine

2.1. **The nilpotent cone.**

- G = complex (semisimple) reductive group, actually we are thinking about $\hat{G}(\mathbb{C})$;
- $\mathfrak{g} = \text{Lie}(G)$

Let $q : \mathfrak{g} \to \mathfrak{g}/G = \mathfrak{h}/W$ be the adjoint quotient map. The *nilpotent cone* is $\mathcal{N} = \bigcup_{0 \in \bar{\mathcal{O}}} \mathcal{O}$

Example 2.1. For $G = \text{GL}_n$, $\mathcal{N} = \{x \in \mathfrak{gl}_n \mid x^n = 0\}$. In this case, one can parametrize the nilpotent elements by

$$\{\text{nilpotent matrices}\}/\text{conjugate} \leftrightarrow \{\text{Jordan normal forms}\} \leftrightarrow \{\text{partitions of } n\}.$$

The same also works for SL_n. Take $n = 2$. There are two conjugacy classes of nilpotent matrices:

$$\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \rightsquigarrow \mathcal{O}_{\lambda_0} = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$$
$$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \rightsquigarrow \mathcal{O}_{\lambda_1} = \{x \in \mathfrak{sl}_2 \mid \text{rank}(x) = 1\}.$$

Example 2.2. Let $G = \text{Sp}_{2n}$. The nilpotent orbits of Sp_{2n} can also be parametrized by certain partitions:

$$\{\text{nilpotent orbits of } \text{Sp}_{2n}(\mathbb{C})\} \leftrightarrow \{\text{partition of } 2n \mid \text{odd partitions occur with even multiplicities}\}.$$

Take $n = 4$, then we have the closure including ordering and dimensions

$$\begin{array}{cc} \mathcal{O}_{[4]} & 8 \\ | & \\ \mathcal{O}_{[2^2]} & 6 \\ | & \\ \mathcal{O}_{[2\ 1^2]} & 4 \\ | & \\ \mathcal{O}_{[1^4]} & 0 \end{array}$$

In general, for $\mathrm{Sp}_{2n}(\mathbb{C})$, let $r_i = |\{j \in \lambda_j = i\}|$ and $s_i = |\{j \mid \lambda_j \geq i\}|$. Then

$$\dim \mathcal{O}_\lambda = 2n^2 + n - \frac{1}{2}\sum s_i^2 - \frac{1}{2}\sum_{\text{odd } i} r_i.$$

We have $\mathcal{O}_{\lambda_1} \leq \overline{\mathcal{O}_{\lambda_2}}$ if and only if $\lambda_1 \leq \lambda_2$ under dominance ordering.

Proposition 2.1. *(1) $\mathcal{N}$ is irreducible, reduced and normal;*
(2) G acts on $\mathcal{N}$ by conjugation, with finitely many orbits, each of which has even dimension.

2.2. **Springer resolution.**

Definition. The *Springer resolution* is the projection to the first factor

$$\pi : \tilde{N} = \{(x, B) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\} \to \mathcal{N}$$

where $\mathcal{B}$ is the variety of Borel subalgebras of $\mathfrak{g}$.

The *Springer fiber* of $x \in \mathcal{N}$ is $\pi_s^{-1}(X)$. For $x \in \mathcal{O}_\lambda$, set $F_\lambda = \pi_s^{-1}(x)$.

Example 2.3. Let $G = \mathrm{SL}_2(\mathbb{C})$. For $x \neq 0$, $\pi_s^{-1}(X) = \{*\}$. For $x = 0$, $\pi_s^{-1}(0) = \mathcal{B} = \mathbb{P}^1(\mathbb{C})$.

Proposition 2.2. $\dim F_\lambda = \frac{1}{2}\operatorname{codim}(\mathcal{O}_\lambda \subseteq \mathcal{N})$.

Proposition 2.3. $\tilde{N} \cong T^*\mathcal{B} = \{(\mathfrak{b}, v) \in B \times \mathfrak{g}^* \mid v \in \mathfrak{b}^\perp\}$

Proof. Let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the killing form. By Cartan's 2nd criterion, κ is non-degenerate. Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{b}$, consider $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Since $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, $\kappa|_{\mathfrak{n}_-\times\mathfrak{n}_+}$ is also a non-degenerate pairing. $\mathfrak{b}^\perp \subseteq \mathfrak{g}^*$ corresponds to $\mathfrak{n}_+$ and hence

$$\begin{aligned} T^*\mathcal{B} &= \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathfrak{g} \mid x \text{ is nilpotent}\} \\ &= \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathcal{N} \mid x \in \mathfrak{b}\} \\ &= \tilde{N}. \end{aligned}$$

□

2.3. **Steinberg variety.** Let $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ be stratified variety (for example $\mathcal{N} = \bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda$).

Definition. The *conormal space* is $T^*_\Lambda X = \bigcup_{\lambda\in\Lambda} T^*_\lambda X \subseteq T^*X$ where $T^*_\lambda X = \{(x, \xi) \in X_\lambda \times T^*_x X \mid \xi \text{ vanishes on } TX_\lambda\}$

Example 2.4. Consider $\mathbb{C} = \mathbb{C}^\times \cup \{0\}$. Then

$$\begin{aligned} T^*_0 &= \mathbb{C} = \{(0, y) \in \mathbb{C}^2\} \\ T^*_x &= \{(x, 0) \in \mathbb{C}^2\} \text{ for } x \neq 0. \end{aligned}$$

Hence $T^*_\Lambda \mathbb{C} = \{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$.

Proposition 2.4. *(1) $T^*_\Lambda X$ is a closed subvariety of X;*
(2) $\dim T^*_\lambda X = \dim X_\lambda + \operatorname{codim}(X_\lambda \subseteq X) = \dim X$;
(3) Irreducible components of $T^*_\Lambda X$ are in bijection with Λ.

Remark. Intersection pattern of $\overline{T^*_\Lambda X}$ is hard.

Definition. Let H be a group which acts on varieties X and Y on the left and right respectively. The *balanced product* is

$$X \times_H Y = X \times Y/((xh, y) \sim (x, hy)).$$

Remark. $X \times_H Y$ is not always a variety, but for our cases of interest it will be.

Fix a Borel $B \subseteq G$, so $\mathcal{B} = G/B$. Consider

$$\begin{aligned} G \times_B G/B &\cong G/B \times G/B \\ (g, g'B) &\mapsto (gg'B, g'B). \end{aligned}$$

The set of G-orbits on $G \times_B G/B$ is equal to the set of B-orbits on G/B. Therefore, the set of B-orbits are parametrized by the Weyl group W by the Bruhat decomposition, i.e.,

$$\mathcal{B} \times \mathcal{B} = \bigsqcup_{x \in W} \mathcal{O}_x.$$

Definition. The *Steinberg variety* is

$$\mathrm{St} = \{(\mathfrak{b}, \mathfrak{b}', x) \in \mathcal{B} \times \mathcal{B} \times \mathcal{N} \mid x \in \mathfrak{b} \cap \mathfrak{b}'\}.$$

In other words, St is the fiber product

$$\begin{array}{ccc} \mathrm{St} & \longrightarrow & \tilde{\mathcal{N}} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{N}} & \longrightarrow & \mathcal{N} \end{array}.$$

Now by Proposition 2.3, we have

$$\begin{aligned} \mathrm{St} &\subseteq T^*\mathcal{B} \times T^*\mathcal{B} \cong T^*(\mathcal{B} \times \mathcal{B}) \\ ((x_1, \mathfrak{b}_1), (x_2, \mathfrak{b}_2)) &\mapsto (x_1, \mathfrak{b}_1, -x_2, \mathfrak{b}_2). \end{aligned}$$

Proposition 2.5. $\mathrm{St} = \bigcup_{(\mathfrak{b}_1, \mathfrak{b}_2)} T^*_{\mathcal{O}_x, (\mathfrak{b}_1, \mathfrak{b}_2)}(\mathcal{B} \times \mathcal{B}) = \bigcup_{(\mathfrak{b}_1, \mathfrak{b}_2)} (T_{(\mathfrak{b}_1, \mathfrak{b}_2)} \mathcal{O}_x)^{\perp}$.

Corollary 2.1. $\mathrm{St} = \bigsqcup_{w \in W} T^*_{\mathcal{O}_w}(\mathcal{B} \times \mathcal{B})$ *is a conormal space.*

3. Kazhdan-Lusztig I - Alex Bauman

3.1. **Convolution algebras.** Let X_1, X_2 be varieties over k. A *correspondence* from X_1 to X_2 is a closed immersion

$$\begin{array}{ccccc} Z_{12} & \hookrightarrow & X_1 \times_k X_2 & \xrightarrow{p_1^{12}} & X_1 \\ & & \downarrow{\scriptstyle p_2^{12}} & & \\ & & X_1 & & \end{array}.$$

Correspondences induce maps on cohomology:

$$\begin{aligned} H^*(X_1) &\to H^*(X_2) \\ c &\mapsto p^{12}_{2,*}((p_1^{12,*}c) \cup [Z_{12}]). \end{aligned}$$

More generally, we get

$$\begin{aligned} H^*(X_1) \otimes H^*(Z_{12}) &\to H^*(X_2) \\ (c \otimes d) &\mapsto p^{12}_{2,*}(p_1^{12,*}c \cup d). \end{aligned}$$

Suppose X_3 is another variety, $Z_{23} \subseteq X_2 \times X_3$, set

$$Z_{12} \circ Z_{23} = Z_{13} = \{(z_1, z_3) \in Z_1 \times_{Z_3} \mid \exists z_2 \in Z_2 \text{ s.t. } (z_1, z_2) \in Z_1 \times Z_2, (z_2, z_3) \in Z_2 \times Z_3\}.$$

The *convolution product in homology* is

$$\begin{aligned} H(Z_{12}) \times H(Z_{23}) &\to H(Z_{12}) \\ (c_{12}, c_{23}) &\mapsto p_{13,*}(p_{12}^* c_{12} \cap p_{23}^* c_{23}). \end{aligned}$$

We have factorization

$$\begin{array}{ccc} H(X_1)\otimes H(Z_{12})\otimes H(Z_{23}) & \longrightarrow & H(X_3) \\ \downarrow & \nearrow & \\ H(X_1)\otimes H(Z_{12}) & & \end{array}.$$

Example 3.1. (1) For $X = X_1 = X_2 = X_3$ finite set, $Z_{ij} = X\times X$, we can identify $H(X\times X)$ as functions $X \times X \to k$. The product is given by

$$(f * g)(x, y) = \sum_{z\in X} f(x, z)g(z, y).$$

One sees that $H(X \times X) \cong M_{*X}(k)$ and the product is matrix multiplication.

(2) Say $\tilde{X}$ is smooth with $\tilde{X} \to X$ is proper. For $X_1 = X_2 = X_3 = \tilde{X}$, $Z_{ij} = \tilde{X} \times_X \tilde{X} \subseteq \tilde{X} \times_k \tilde{X}$. Then $H(\tilde{X} \times \tilde{X})$ has a ring structure.

3.2. **Equivariant Grothendieck group.** Let G be an algebraic group action on some variety X, both defined over k. Then we have

$$\begin{array}{ccc} G\times X & \xrightarrow{p_2} & X \\ \downarrow{\scriptstyle\sigma} & & \\ X & & \end{array}, \quad \begin{array}{ccc} G\times G\times X & \xrightarrow{p_{23}} & X \\ \downarrow{\scriptstyle m} & & \\ G & & \end{array}.$$

An *equivariant sheaf* is a sheaf $\mathcal{F}$ on X together with an isomorphism

$$\varphi : \sigma^*\mathcal{F} \to p_2^*\mathcal{F}$$

such that $(p_{23}^*\varphi) \circ (\mathrm{id} \times \sigma)^*\varphi = (m \times \mathrm{id})^*\varphi$.

Example 3.2. If G is affine, G-action on X is trivial, then equivariant sheaf is just a sheaf of $\mathcal{O}[G] = \mathcal{O}_X \otimes \mathcal{O}_G$-modules.

Definition. We define the *K-group of G-equivariant coherent sheaves* on X as

$$K^G(X) := \frac{\mathbb{Z}\{[\mathcal{F}] \mid \mathcal{F} = \text{ coherent equivariant sheaf on } Z\}}{\{[\mathcal{G}] - [\mathcal{F}] - [\mathcal{H}] \mid \exists \text{ SES } \mathcal{F} \to \mathcal{G} \to \mathcal{H} \text{ of } G\text{-sheaves}\}}.$$

Remark. K^G behaves steadily to Borel-Moore homology, so K^G is equipped with a convolution algebra structure.

Remark. G-equivariant sheaves are the same as sheaves on the quotient stack $[X/G]$.

3.3. **Kazhdan-Lusztig isomorphism.** Let F be a local field and $\mathbf{G}/F$ split reductive group over F. Let $I \subseteq \mathbf{G}(F)$ be the Iwahori. Recall that the Iwahori-Matsumoto extended Hecke algebra $\mathcal{H}_{\text{ext}}$ is an algebra over $\mathbb{Z}[v^{\pm 1}]$ with

$$\mathcal{H}_{\text{ext}} \otimes_{\mathbb{Z}[v^{\pm1}], v\mapsto q^{\frac{1}{2}}} \mathbb{C} = \mathcal{H}(I\backslash\mathbf{G}(F)/I, \mathbb{C}).$$

Theorem 3.1 (Kazhdan-Lusztig)**.** *There is a commutative diagram*

$$\begin{array}{ccc} \mathcal{H}_{ext} & \xrightarrow{\sim} & K^{\hat{G}\times\mathbb{G}_m}(\mathrm{St}) \\ {\scriptstyle v=1}\downarrow & & \downarrow{\scriptstyle forgetful} \\ Z[W_{ext}] & \longrightarrow & K^{\hat{G}}[\mathrm{St}] \end{array}.$$

where the action of $\hat{G} \times \mathbb{G}_m$ is giving by adjoint action and scaling in $\mathcal{N}$.

Remark. Consider the diagonal $\tilde{\mathcal{N}} \subseteq \mathrm{St} = \tilde{N} \times_{\mathcal{N}} \tilde{\mathcal{N}}$. Using $\tilde{N} \cong T^*\mathcal{B}$, we have $K^{\hat{G}\times\mathbb{G}_m}(\tilde{\mathcal{N}}) = K^{\hat{G}\times\mathbb{G}_m}(\mathcal{B}) = K^{\hat{B}\times\mathbb{G}_m}(*) = \mathrm{Rep}(\hat{B}) \otimes \mathrm{Rep}(\mathbb{G}_m) = \mathbb{Z}[X^\vee] \otimes \mathbb{Z}[v^{\pm 1}] \cong R_T$. So we have $\mathbb{Z}[v^{\pm 1}][X^\vee]$ sitting inside $\mathcal{H}_{\mathrm{ext}}$. Moreover, $K^{\hat{G}\times\mathbb{G}_m}(*) = (K^{\hat{B}\times\mathbb{G}_m}(*))^{W_{\mathrm{fin}}} = (\mathbb{Z}[v^{\pm 1}][X^\vee])^{W_{\mathrm{fin}}}$ gives the Bernstein center.

4. Kazhdan-Lusztig II - Calvin Yost-Wolff

4.1. Recap.

- Alex introduced the convolution product. In particular, form the commutative diagram

$$\begin{array}{ccccc} & & \tilde{\mathcal{N}}_1 \times_{\mathcal{N}} \tilde{\mathcal{N}}_2 \times_{\mathcal{N}} \tilde{\mathcal{N}}_3 & & \\ & \swarrow^{p_{12}} & \downarrow{\scriptstyle p_{13}} & \searrow^{p_{23}} & \\ \tilde{\mathcal{N}}_1 \times_{\mathcal{N}} \tilde{\mathcal{N}}_2 & & \tilde{\mathcal{N}}_1 \times_{\mathcal{N}} \tilde{\mathcal{N}}_3 & & \tilde{\mathcal{N}}_2 \times_{\mathcal{N}} \tilde{\mathcal{N}}_3 \end{array}$$

  we get the convolution product given by

$$\mathcal{F}_{12} * \mathcal{F}_{23} = p_{13,*}(p_{12}^*\mathcal{F}_{12} \otimes p_{23}^*\mathcal{F}_{23}).$$

- Let $Z = \tilde{\mathcal{N}}_1 \times_{\mathcal{N}} \tilde{\mathcal{N}}_2$, then there is an action $K^{G\times\mathbb{G}_m}(Z) \curvearrowright K^{G\times\mathbb{G}_m}(\tilde{N})$ given by

$$\mathcal{F} \cdot \mathcal{G} = p_{1,*}(\mathcal{F} \otimes p_2^*\mathcal{G}).$$

There are also two questions remain.

- What equivariant sheaves correspond to which elements in $K^{G\times\mathbb{G}_m}(\mathrm{St})$?
- What is a the approach to Kazhdan-Lusztig isomorphism.

Example 4.1. We have isomorphisms

$$\begin{array}{ccccc} K^G(\tilde{N}) & \longrightarrow & K^G(G/B) & \longrightarrow & K^B(*) = \mathbb{C}[X] \\ & & & & . \\ e^\lambda & \longleftarrow & G \times^B \lambda := L(\lambda) & \longleftarrow & \lambda \end{array}$$

Another way to is this is via localization. For nice $T \subseteq G \curvearrowright X$, $K^G(X) = [K^T(X)]^{W_{\mathrm{fin}}}$. For $X = G/B$, $X^T = \{wB/B \mid w \in W_{\mathrm{fin}}\}$. λ should correspond a equivariant vector bundle whose fiber at wB/B is λ^w.

These $L(\lambda)$'s are well-understood and important later. Let s be a simple transposition and α be the corresponding root. Let $\pi_s : G/B \to G/P_s$ be the natural projection. Then we have the following "Weyl character formula" type of formula

$$\pi_s^*\pi_{s,*}e^\lambda = \frac{e^{\lambda+\frac{\alpha}{2}} - e^{s(\lambda)-\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}}.$$

4.2. Strategy for understanding Kazhdan-Lusztig isomorphism.

1. On the Bernstein generators of $\mathcal{H}$, define a map to $K^{G\times\mathbb{G}_m}(\mathrm{St})$
   - finite part: $T_s \mapsto Q_s$ (to be defined later),
   - lattice part: $\theta_\lambda \mapsto \mathcal{O}_\lambda =$ pull-back of $\mathcal{O}(\lambda)$ along $T^*\mathcal{B} \xrightarrow{\mathrm{diag}} \mathrm{St}$.
2. Define the anti-spherical module M of $\mathcal{H}$ and study $\mathcal{H} \curvearrowright M$.
3. Study the action $K^{G\times\mathbb{G}_m}(Z) \curvearrowright K^{G\times\mathbb{G}_m}(\tilde{N})$.
4. Match the actions above at the level of generators introduced in (1). Since both actions are faithful, this implies the Kazhdan-Lusztig isomorphism.

4.3. **Anti-spherical module.** To prove Kazhdan-Lusztig isomorphism, we are going to show that $K^{G\times\mathbb{G}_m}(Z) \curvearrowright K^{G\times\mathbb{G}_m}(\tilde{\mathcal{N}}) = \mathbb{Z}[v,v^{-1}][X]$ realize an action of Iwahori Hecke algebra on $\mathbb{Z}[v,v^{-1}][X]$.

Recall that Elad defined

$$\mathcal{H} := \mathbb{Z}[v,v^{-1}][\{T_s\},\{\theta_\lambda\}]/\sim$$

where the relation $\sim$ consists of

- quadratic relation: $(T_s+v)(T_s-v^{-1})=0$,
- BZ relation: $T_s\theta_\lambda - \theta_{s(\lambda)}T_s = (v-v^{-1})\left(\frac{\theta_\lambda-\theta_{s(\lambda)}}{1-\theta_{-\alpha}}\right)$.

Recall the sign representation $\mathcal{H}_{\mathrm{fin}} \curvearrowright \mathbb{Z}[v,v^{-1}]$ where all T_s acts as v^{-1}. The *anti-spherical module* is $M := \mathcal{H}\otimes_{\mathcal{H}_{\mathrm{fin}}}\mathbb{Z}[v,v^{-1}]$ equipped with $\mathcal{H}$-action via the first factor. M has a $\mathbb{Z}[v,v^{-1}]$-basis given by $\theta_\lambda\otimes 1 =: [\theta_\lambda]$. Then these $[\theta_\lambda]$'s satisfy the relation

- $\theta_\mu\cdot[\theta_\lambda] = [\theta_{\lambda+\mu}]$, and
- (from BZ relation) $(T_s+v)[\theta_\lambda] = (v[\theta_{-\alpha}]-v^{-1})\left(\frac{[\theta_\lambda]-[\theta_{s(\lambda)}]}{1-[\theta_{-\alpha}]}\right)$.

Moreover, M is a faithful $\mathcal{H}$-module.

4.4. **Explicit computation of $K^{G\times\mathbb{G}_m}(Z)\curvearrowright K^{G\times\mathbb{G}_m}(\tilde{N})$.**

Example 4.2. Let first look at the case where $G=\mathrm{SL}_2$. Recall from Alex's talk that we have $Z = T^*\mathbb{P}^1\cup(\mathbb{P}^1\times\mathbb{P}^1\backslash\Delta)$. The diagonal part is easy to handle:

$$e^\mu_\Delta\cdot e^\lambda = e^{\mu+\lambda}.$$

The complement is more difficult. Let's compute the action of $Q_s = \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-2,0) = p_1^*\Omega_{\mathbb{P}^1/*}$ where p_1,p_2 are the projections:

$$\begin{array}{ccc} \mathbb{P}^1\times\mathbb{P}^1 & \xrightarrow{p_2} & \mathbb{P}^1 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 & \xrightarrow{\pi} & \{*\} \end{array}.$$

We want to compute $p_{1,*}(Q_s\otimes p_2^*e^\lambda)$:

$$\begin{aligned} p_{1,*}(Q_s\otimes p_2^*e^\lambda) &= [\Omega_{\mathbb{P}^1}]\otimes p_{1,*}p_2^*(e^\lambda) && \text{(projection formula)}\\ &= [\Omega_{\mathbb{P}^1}]\cdot\left(\frac{e^{\lambda+\frac{\alpha}{2}}-e^{-\lambda-\frac{\alpha}{2}}}{e^{\frac{\alpha}{2}}-e^{-\frac{\alpha}{2}}}\right) && \text{(base change formula).} \end{aligned}$$

Now use the following fact: let $\pi : E\to X$ be a vector bundle, $\mathcal{F}$ a quasi-coherent sheaf on X, $\mathcal{G}$ be a free sheaf on X and $i : X\to E$ be the zero section. Then we have

$$i_*\mathcal{F}\otimes\pi^*\mathcal{G} = i_*(\mathcal{F}\otimes\mathcal{G}).$$

Apply this fact to $\tilde{\mathcal{N}}\times\tilde{\mathcal{N}}\to\mathbb{P}^1\times\mathbb{P}^1$. We get $[i_*\Omega_{\mathbb{P}^1}] = (e^{-\alpha}-v^{-2}\cdot e^0)$ via Koszul resolution. Therefore, we get

$$(T_s+v) = v\cdot Q_s.$$

Now let's generalize this method. For general G, the components of Z for simple transposition pairs

$$\begin{array}{ccccccc} \bar{\mathcal{O}}_s & \longrightarrow & G/B & & \tilde{\mathcal{Z}}_s & \longrightarrow & \tilde{N}_s \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle \pi_s} & \xleftarrow{q} & \downarrow & & \downarrow \\ G/B & \xrightarrow{\pi_s} & G/P_s & & \tilde{\mathcal{N}}_s & \longrightarrow & T^*G/P_s \end{array}$$

where $\tilde{N}_s = \{(e, B) \mid e \in \text{unipotent radical of } \pi(B)\}$, $\bar{\mathcal{O}}_s \subseteq \mathcal{B} \times \mathcal{B}$ is the G-orbit consisting of pairs of flags in relative position s. As earlier in the case of SL_2, the same trick implies a similar result for $Q_s = q^*\Omega_{\bar{\mathcal{O}}_s/\mathcal{B}}$:

$$p_{1,*}(Q_s \otimes p_2^*\mathcal{O}(\lambda)) = \left(\frac{e^\lambda - e^{s(\lambda)-\alpha}}{1 - e^{-\alpha}}\right)[i_*\mathcal{O}_{\tilde{\mathcal{N}}_s}(-\alpha)]$$

where $i : \tilde{\mathcal{N}}_s \hookrightarrow \tilde{\mathcal{N}}$ is the inclusion. Again, one needs to express $[i_*\mathcal{O}_{\tilde{\mathcal{N}}_s}(-\alpha)] \in K^{G\times\mathbb{G}_m}(\tilde{\mathcal{N}})$ in the basis of line bundles. This can be done using a Koszul resolution for the short exact sequence of B-modules

$$0 \to \mathfrak{p}_s/\mathfrak{b} \to \mathfrak{g}/\mathfrak{g} \to \mathfrak{g}/\mathfrak{p}_s \to 0.$$

4.5. **Completing the proof.** There are two things remained to be showed:

(1) Q_s and e^λ_Δ generate $K^{G\times\mathbb{G}_m}(Z)$, and
(2) the action $K^{G\times\mathbb{G}_m}(Z) \curvearrowright K^{G\times\mathbb{G}_m}(\tilde{N})$ is faithful.

We already have the lower bound of $\dim_{K^{G\times\mathbb{G}_m}(\tilde{N})} K^{G\times\mathbb{G}_m}(Z)$ is equal to $|W_{\text{fin}}|$. Let's compute the upper bound. There are two ways to get upper bound.

(1) Localization. We can embed $K^{G\times\mathbb{G}_m}(Z) \hookrightarrow (K^{T\times\mathbb{G}_m}(Z^{T\times\mathbb{G}_m}))^W$ with $Z^{T\times\mathbb{G}_m} = \{0, w_1B/B, w_2B/B\}$.
(2) Something like MV sequence?

5. A crash course on derived categories and perverse sheaves - Robert Cass

5.1. **Derived functors.** Let $\mathcal{A}$ be an abelian category. Many functors of abelian categories don't preserve exact sequences.

Example 5.1. Let $\mathcal{A} = \mathrm{Sh}(X, R)$, the global section functor $\Gamma(X, -)$ is left exact but not exact.

Derived functors remedy the failure of exactness. For example, we can set $\Gamma(X, -) = H^0(X, -)$ and define a sequence of functors $H^i(X, -)$. Then short exact sequence

$$0 \to A \to B \to C \to 0$$

gives rise to long exact sequence

$$\begin{array}{c} \cdots \longrightarrow H^i(X,A) \longrightarrow H^i(X,B) \longrightarrow H^i(X,C) \\ \xrightarrow{\ \partial\ } \\ H^{i+1}(X,A) \longrightarrow H^{i+1}(X,B) \longrightarrow H^{i+1}(X,C) \longrightarrow \cdots \end{array}.$$

For now, derived functor is viewed as a sequence of functors. To study perverse sheaves, we would like to package the sequence into one single functor.

From now on, we will assume $\mathcal{A}$ is a Grothendieck abelian category. Let $\mathrm{Ch}^+(\mathcal{A})$ be the category of bounded below chain complexes. A *quasi-isomorphism* is a chain map $f : C_\bullet \to D_\bullet$ such that it induces isomorphisms $H^i(C_\bullet) \cong H^i(D_\bullet)$.

The *derived category* $D^+(\mathcal{A})$ is the localization of $\mathrm{Ch}^+(\mathcal{A})$ at quasi-isomorphisms. In particular, there is a map $\mathrm{Ch}^+(\mathcal{A}) \to D^+(\mathcal{A})$ which is universal for the maps in which quasi-isomorphisms are sent to isomorphisms.

If $C_\bullet, D_\bullet$ happen to chain complex of injective objects, then

$$\mathrm{Hom}_{D^+(\mathcal{A})}(C_\bullet, D_\bullet) = \mathrm{Hom}_{K^+(\mathrm{Inj}(\mathcal{A}))}(C_\bullet, D_\bullet).$$

where the maps of chain complexes are taken up to homotopy.

If $F : \mathcal{A} \to \mathcal{A}'$ is left exact, there is a derived functor $RF : D^+(\mathcal{A}) \to D^+(\mathcal{A}')$ such that

$$H^i(RF(\mathcal{C}_\bullet)) = R^iF(C_\bullet).$$

RF is characterized by the following diagram:

$$\begin{array}{ccc} \mathrm{Ch}^+(\mathcal{A}) & \xrightarrow{F} & \mathrm{Ch}^+(\mathcal{A}') \\ \downarrow q & \swarrow\!\!\!\swarrow & \downarrow q' \\ D^+(\mathcal{A}) & \xrightarrow{RF} & D^+(\mathcal{A}') \end{array}$$

where "$\Rightarrow$" is a natrual transformation $q' \circ F \Rightarrow RF \circ q$.

$D^+(\mathcal{A})$ is triangulated, i.e., there is an auto-equivalence [1] given by $(C_\bullet[1])_i = C_{i+1}$. There is also a notion of exact triangles. If $M_1 \to M_2 \to M_3$ is exact in $D^+(\mathcal{A})$, then we get long exact sequence of $H^i(M_j)$'s.

5.2. **Perverse sheaves.** Let X be a scheme of finite type over a field k. Fix a prime $l \neq \mathrm{char}(k)$. Let $\mathrm{Sh}(X) := \mathrm{Sh}_{\text{ét}}(X, \bar{\mathbb{Q}}_l)$.

There is a nice class of sheaves called *local system*. These are sheaves $\mathcal{L} \in \mathrm{Sh}(X)$ such that there exists an étale cover $f : Y \to X$ for which $f^*\mathcal{L}$ is the sheaf associated to a constant $\bar{\mathbb{Q}}_l$-vector space of finite dimension.

A sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ is *constructible* if there exists a finite stratification of $X = \bigsqcup X_i$ with X_i locally closed such that $\mathcal{F}|_{X_i}$ is a local system.

Let $f : X \to Y$ be a map of schemes. Then there are adjoint functors

$$\mathrm{Sh}(X) \underset{f_*}{\overset{f^*}{\longrightarrow}} \mathrm{Sh}(Y).$$

f^* is exact, while f_* is only left exact. Let $D^b(X)$ be the bounded derived category of constructible sheaves. Therefore, we get derived functors

$$D^b(Y) \underset{f^*}{\overset{Rf_*}{\longrightarrow}} D^b(X)$$
$$D^b(X) \underset{Rf_!}{\overset{f^!}{\longrightarrow}} D^b(Y).$$

We also have a derived $\otimes$ and $\underline{\mathrm{Hom}}$.

For $f : X \to \operatorname{Spec} k$. The *dualizing sheaf* is $\omega_X = f^!\bar{\mathbb{Q}}_l$. Then we have the Verdier duality

$$\mathbb{D} : D^b(X) \to D^{b(X)},$$
$$\mathcal{F} \mapsto \underline{R\,\mathrm{Hom}}(\mathcal{F}, \omega_X).$$

If X is smooth, and $\mathcal{L}$ is a local system on X. Then $\mathcal{L}[\dim X] \in D^b(X)$ is an example of a perverse sheaf.

Define three subcategories of $D^b(X)$.

- ${}^pD^b(X)^{\leq 0}$ consists of all $\mathcal{F} \in D^b(X)$ such that there exists a stratification $X = \bigsqcup X_i$ with
$$H^n(\mathcal{F}|_{X_i}) = \begin{cases} 0 & \text{if } n > -\dim X_i, \\ \text{a local system} & \text{if } n \leq -\dim X_i. \end{cases}$$
- ${}^pD^b(X)^{\geq 0}$ consists of all $\mathcal{F} \in D^b(X)$ such that $\mathbb{D}(\mathcal{F}) \in {}^pD^b(X)^{\leq 0}$.
- $\mathrm{Perv}(X)$ is the intersection of ${}^pD^b(X)^{\geq 0}$ and ${}^pD^b(X)^{\leq 0}$.

$\mathrm{Perv}(X)$ is an abelian category (heart of t-structure). Objects in $\mathrm{Perv}(X)$ have finite length.

There is a cohomology function

$${}^pH^i(-) : D^b(X) \to \mathrm{Perv}(X)$$

sending exact triangles to long exact triangles.

5.3. **Classification of irreducible perverse sheaves.** Suppose $j : X \to Y$ is a locally closed immersion. The *intermediate extension* is

$$\begin{aligned} j_{!*} : \mathrm{Perv}(X) &\to \mathrm{Perv}(Y) \\ \mathcal{F} &\mapsto \mathrm{Im}({}^pH^0(Rj_!(\mathcal{F})) \to {}^pH^0(Rj_*(\mathcal{F}))). \end{aligned}$$

If $\mathcal{F}$ is simple, $j_{!*}$ is simple (as perverse sheaf). If j is a closed immersion, then $j_* = j_! = j_{!*}$.

Theorem 5.1. *Every simple perverse sheaf $\mathcal{F}$ on X is of the form $j_{!*}(\mathcal{L}[\dim U])$ for $U \subseteq X$ smooth and irreducible and $\mathcal{L}$ simple local system on U.*

Example 5.2. Let $X = \mathbb{P}^1$.

(1) $\bar{\mathbb{Q}}_l[1]$ is a simple perverse sheaf on $\mathbb{P}^1$.
(2) Let $i : \{0\} \to \mathbb{P}^1$ be the inclusion. $i_{!*}(\bar{\mathbb{Q}}_l[0]) = i_*(\bar{\mathbb{Q}}_l[0])$ is simple (constant sheaf supported on a point).
(3) Let $j : \mathbb{A}^1 \to \mathbb{P}^1$. $j_{!*}(\bar{\mathbb{Q}}_l[1]) \cong \bar{\mathbb{Q}}_l[1]$.
(4) Let $\mathcal{L}$ be a local system on $\mathbb{A}^1$ which does no extend to a local system on $\mathbb{P}^1$. $j_{!*}(\mathcal{L}[1])$ is not a local system shifted by 1 because

$$H^{-1}(i_\infty^*(j_{!*}(\mathcal{L}[1]))) = 0$$

but for local system this should be zero (if $\mathcal{L}$ is Artin-Schreier because $j_{!*}\mathcal{L}[1] = j_!\mathcal{L}[1]$).

6. Geometric Satake equivalence - Robert Cass

6.1. **Equivariant perverse sheaves.** Let $k = \bar{k}$ and l a prime not equal to $\mathrm{char}(k)$. Assume that an affine algebraic k-group G acts on X of finite type, satisfying the following assumptions:

(1) there are finitely many orbits;
(2) The stabilizers are connected.

$\mathcal{F} \in \mathrm{Perv}(X)$ is *G-equivariant* if

$$p^*\mathcal{F} \cong \pi^*\mathcal{F},$$

where $p : G \times X \to X$ is the projection the second factor and $\pi : G \times X \to X$ is the action map. The category $\mathrm{Perv}_G(X) \subseteq \mathrm{Perv}(X)$ of G-equivariant perverse sheaves is a full abelian subcategory stable under subquotient.

The simple objects in $\mathrm{Perv}_G(X)$ can be constructed as follows. Take $j : U \hookrightarrow X$ to be a G-orbit. Let

$$\mathrm{IC}_U := j_{!*}\bar{\mathbb{Q}}_l[\dim U].$$

6.2. **Derived equivariant sheaves.** Let $D_G(X)$ be the bounded, constructible equivariant derived category. This is not a subcategory of $D^b(X)$, but we still has a six-functions formalism for G-equivariant $X \to Y$. One also has the cohomology functor

$${}^pH^i(-) : D_G(X) \to \mathrm{Perv}_G(X)$$

that is jointly conservative.

6.3. **Geometric Satake equivalence.**

6.3.1. *Classical Satake isomorphism.* Let F be a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p(\!(t)\!)$. Let G be a split connected reductive group. Let $\mathcal{O} = \mathcal{O}_F$ be the ring of integers of F.

The classical Satake isomorphism is an isomorphism

$$\mathcal{H}(G(F) /\!\!/ G(\mathcal{O})) \xrightarrow{\sim} K(\mathrm{Rep}_{\bar{\mathbb{Q}}_l}(\hat{G})) = K^{\hat{G}}(\mathrm{Spec}\,\bar{\mathbb{Q}}_l).$$

6.3.2. *Geometric Satake equivalence.* The geometric Satake equivalence is an equivalence of categories whose "sheaf-to-function" dictionary recovers the classical Satake equivalence.

$$\begin{array}{ccc} \mathrm{Perv}_{L^+G}(G) & \xrightarrow{\sim} & \mathrm{Perv}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_l}(\hat{G}) \\ \downarrow K & & \downarrow K \\ \mathcal{H}(G(F) /\!/ G(\mathcal{O})) & \longrightarrow & K(\mathrm{Rep}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_l}(\hat{G})) \end{array}.$$

6.3.3. *Loop groups.* Let $k = \bar{\mathbb{F}}_q$, $F = \bar{\mathbb{F}}_q(\!(t)\!)$, $\mathcal{O} = \bar{\mathbb{F}}_q[\![t]\!]$. Define the following functors on k-algebras

$$\begin{aligned} k\text{-alg} &\to \mathrm{Sets} \\ LG : R &\mapsto G(R(\!(t)\!)), \\ L^+G : R &\mapsto G(R[\![t]\!]). \end{aligned}$$

Then the affine Grassmannian $\mathrm{Gr} = LG/L^+G$ is equipped with L^+-action.

Gr is ind-projective, i.e., increasing union of projective k-schemes. L^+G is a group scheme. Therefore we can study $\mathrm{Perv}_{L^+G}(\mathrm{Gr})$.

First we study the orbit of L^+G action on Gr. Let $T \subseteq B$ be a maximal torus in a Borel B. $\mu \in X_*(T)$ is *dominant* if $\langle \alpha.\mu \rangle \geq 0$ for any positive roots. Let $X_*(T)^+$ be the set of dominant cocharacters.

The Cartan decomposition writes

$$G(F) = \bigcup_{\mu \in X_*(T)^+} G(\mathcal{O}) \cdot \mu(t) \cdot G(\mathcal{O}).$$

L^+G-orbits in Gr are indexed by $X_*(T)^+$. The orbit Gr^μ is smooth and finite-dimensional. It turns out that

$$\bar{\mathrm{Gr}}^\mu = \mathrm{Gr}^{\leq \mu} = \bigsqcup_{\nu \leq \mu} \mathrm{Gr}^\nu.$$

let $j_\mu : \mathrm{Gr}^{\leq \mu} \to \mathrm{Gr}$ be the natural embedding. Then

$$\mathrm{IC}_\mu = (j_\mu)_{!*} \bar{\mathbb{Q}}_l [\dim \mathrm{Gr}^{\leq \mu}]$$

exhaust the simple objects.

6.3.4. *Convoltuion.* Consider

$$\mathrm{Gr} \times \mathrm{Gr} \xleftarrow{(\mathrm{quotient} \times \mathrm{id})} LG \times \mathrm{Gr} \xrightarrow{q} LG \times^{L^+G} \mathrm{Gr} \xrightarrow{m} \mathrm{Gr}.$$

Let $p_1, p_2 : \mathrm{Gr} \times \mathrm{Gr} \to \mathrm{Gr}$ be the projection on the i-th factor. For $\mathcal{F}_1, \mathcal{F}_2 \in \mathrm{Perv}_{L^+G}(\mathrm{Gr})$, define

$$\mathcal{F}_1 \boxtimes \mathcal{F}_2 = p_1^* \mathcal{F}_1 \otimes p_2^* \mathcal{F}_2.$$

There is a unique $\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2 \in \mathrm{Perv}_{L^+G}(LG \times^{L^+G} \mathrm{Gr})$ such that

$$q^*(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2) \cong p^*(\mathcal{F}_1 \boxtimes \mathcal{F}_2).$$

We define

$$\mathcal{F}_1 * \mathcal{F}_2 = m_!(\mathcal{F}_1 \tilde{\boxtimes} \mathcal{F}_2).$$

Proposition 6.1. *(1) $\mathcal{F}_1 * \mathcal{F}_2$ is perverse (semi-small, nearby cycles, etc);*
(2) This is symmetric, i.e., there are natural maps $\mathcal{F}_1 * \mathcal{F}_2 \cong \mathcal{F}_2 * \mathcal{F}_1$.

(3) The global section functor

$$\bigoplus_i R\Gamma^i(-) : \mathrm{Perv}_{L^+G}(\mathrm{Gr}) \to \mathrm{Vect}_{\bar{\mathbb{Q}}_l}$$

is exact, faithful ⊗-functor that fits into the commutative diagram

$$\begin{array}{ccc} \mathrm{Perv}_{L^+G}(G) & \xrightarrow{\textit{Satake}} & \mathrm{Rep}^{alg}_{\bar{\mathbb{Q}}_l}(\hat{G}) \\ {\scriptstyle \bigoplus R^i\Gamma}\downarrow & \swarrow {\scriptstyle \textit{forgetful}} & \\ \mathrm{Vect}_{\bar{\mathbb{Q}}_l} & & \end{array} .$$

6.3.5. *Simple objects.* The category $\mathrm{Perv}_{L^+G}(\mathrm{Gr})$ is Tannakian. Therefore the Tannakian formalism implies

$$\mathrm{Perv}_{L^+G}(\mathrm{Gr}) \cong \mathrm{Rep}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_l}(H)$$

for some algebraic group H over $\bar{\mathbb{Q}}_l$.

Let $\hat{T} \subseteq \hat{G}$ be the dual torus, $V \in \mathrm{Rep}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_l}(\hat{G})$,

$$V = \bigoplus_{\mu \in X^*(\hat{T})} V_\mu.$$

FOr each $\mu \in X^*(T) \cong X_*(\hat{T})$, there is a unique simple $\hat{G}$ representation $L(\mu)$ of highest weight μ such that

$$\dim L(\mu)_\mu = 1$$
$$L(\mu)_\nu = 0 \text{ unless } \nu \leq mu.$$

Under the geometric Satake equivalence,

$$\mathrm{IC}_\mu \mapsto L(\mu).$$

7. Gaitsgory's central sheaves - Sean Cotner

7.1. **Introduction.** Let k be an algebraically closed field and G a connected reductive group over k. Let $T \subseteq B$ be a Borel-torus pair. Let $K = k((t)), \mathcal{O} = k[\![t]\!]$ and $I \subseteq G(\mathcal{O})$ be the Iwahori subgroup.

Last time we talked about the geometric Satake equivalence

$$\begin{array}{ccc} \mathrm{Perv}_{L^+G}(G) & \xrightarrow{\sim} & \mathrm{Perv}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_l}(\hat{G}) \\ \downarrow{\scriptstyle K} & & \downarrow{\scriptstyle K} \\ \mathcal{H}(G(F) /\!\!/ G(\mathcal{O})) & \longrightarrow & K(\mathrm{Rep}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_l}(\hat{G})). \end{array}$$

Today we will geometrize Bernstein's description of $Z(\mathcal{H}^{\mathrm{aff}})$.

Theorem 7.1 (Bernstein)**.** $Z(\mathcal{H}^{\mathit{aff}}) \cong \mathbb{C}[X_*(T)]^{W_f} \cong \mathbb{C} \otimes_{\mathbb{Z}} K(\mathrm{Rep}\,\hat{G})$.

Let $\mathrm{Gr} = G(K)/G(\mathcal{O})$ and $Fl = G(K)/I$. Then

$$\mathcal{H}^{\mathrm{aff}} = \mathrm{Fun}^{\mathrm{cts}}_I(Fl, \mathbb{C}), \mathcal{H}^{\mathrm{sph}}\,\mathrm{Fun}_{G(\mathcal{O})} = \mathrm{Fun}^{\mathrm{ss}}_{G(\mathcal{O})}(\mathrm{Gr}, \mathbb{C}).$$

In the classical setting, there is a commutative diagram

$$\begin{array}{ccc} Z(\mathcal{H}^{\mathrm{aff}}) & \xrightarrow{\int_{G(\mathcal{O})/I}} & \mathbb{C} \otimes_{\mathbb{Z}} K(\mathrm{Rep}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_l}(\hat{G})) \\ {\scriptstyle Z}\uparrow & \nearrow {\scriptstyle \text{pullback}} & \\ \mathcal{H}^{\mathrm{sph}} & & \end{array}$$

We want to geometrize this picture. We have seen the geometrization

$$\mathcal{H}^{\text{sph}} \rightsquigarrow \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr})$$
$$\mathcal{H}^{\text{aff}} \to \mathrm{Perv}_I(Fl).$$

So the question is to geometrize the center.

Theorem 7.2 (Gaitsgory)**.** *There exists a central functor*

$$Z : \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}) \to \mathrm{Perv}_I(Fl)$$

such that

$$\begin{array}{ccc} D_I^b(Fl) & \xrightarrow{\pi_*} & D_I^b(\mathrm{Gr}) \\ {\scriptstyle Z}\uparrow & \nearrow_{\textit{forgetful}} & \\ \mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}) & & \end{array}$$

and Z induces the data of natural isomorphisms

$$Z(\mathcal{A}) *^I \mathcal{F} \cong \mathcal{F} *^I Z(\mathcal{A}).$$

7.2. **Variants on a theme.** $LG(R) = G(R⸨t⸩)$, $L^+G(R) = G(R[\![t]\!])$, $\mathrm{Gr} = LG/L^+G$, $D_R = \mathrm{Spec}\, R[\![x]\!]$ and $D_R^\circ = \mathrm{Spec}\, R(\!(x)\!)$.

Proposition 7.1. $\mathrm{Gr}(R) = \{(\xi.\beta) \mid \xi$ *is a G-torsor on* $D_R, \beta : \xi|_{D_R^\circ} \cong \xi_0$ *is a trivializtion*$\}$.

Proposition 7.2. *Choose $x \in C(R)$, let $C^0 = C - \{x\}$, then*

$$\mathrm{Gr}(R) = \{(\xi.\beta) \mid \xi \textit{ is a } G\textit{-torsor on } C_R, \beta : \xi|_{C_R^0} \cong \xi_0 \textit{ is a trivializtion}\}$$

Remark (Beauville-Laszlo)**.** A pair (ξ, β) on D_R can be glued to $\xi^\circ_{C_R^\circ}$ to globalize.

7.2.1. *Variant A.* Let x move.

$\mathrm{Gr}_{G,C}(R) = \{(\xi.\beta, x) \mid x \in C(R), \xi$ is a G-torsor on $C_R, \beta : \xi|_{D_R^\circ - \{x\}} \cong \xi^0|_{C_R - \{x\}}$ is a trivializtion$\}$.

7.2.2. *Variant B.* Allow several points.

$\mathrm{Gr}_{G,C}(R) = \{(\xi.\beta, x) \mid x \in C^n(R), \xi$ is a G-torsor on $C_R, \beta : \xi|_{D_R^\circ - \Gamma_x} \cong \xi^0|_{C_R - \Gamma_x}$ is a trivializtion$\}$.

Remark. When $n = 2$,

$$\mathrm{Gr}_{G,C^2}|_{C^2 - \Delta_C} \cong \mathrm{Gr}_{G,C} \times \mathrm{Gr}_{G,C,C^2-\Delta_C}, \mathrm{Gr}_{G,C^2}|_{\Delta_C} \cong \mathrm{Gr}_{G,C}.$$

But what about semi-continuity of fibber dimension?

$$\overline{\mathrm{Gr}}^{\le\lambda} \times \overline{\mathrm{Gr}}^{\le\mu} \le \overline{\bar{Gr}}^{\le\lambda+\mu}, LG \tilde{\times} \mathrm{Gr} = \mathrm{Gr} \tilde{\times} \mathrm{Gr}.$$

Proposition 7.3.

$$(\mathrm{Gr} \tilde{\times} \mathrm{Gr})(R) = \{(\xi_i, \beta_i, x) \mid x \in C(R), \xi_i \textit{ are } G\textit{-torsors on } C_R, \beta : \xi^i|_{D_R^\circ - \{0\}} \cong \xi^{i+1}|_{C_R - \{0\}} \textit{ is a trivializtion}\}$$
$$(\mathrm{Gr}_{G,C} \tilde{\times} \mathrm{Gr})_{G,C}(R) = \{(\xi_i, \beta_i, x) \mid x \in C(R), \xi_i \textit{ are } G\textit{-torsors on } C_R, \beta : \xi^i|_{D_R^\circ - \{x_i\}} \cong \xi^{i+1}|_{C_R - \{x_i\}} \textit{ is a trivializtion}$$

There is a natural map

$$\mu : \mathrm{Gr}_{G,C} \tilde{\times} \mathrm{Gr}_{G,C} \cong \mathrm{Gr}_{G,C^2}$$
$$(\xi^i, \beta_i, x_i) \mapsto (\xi^2|_{C_R - \Gamma_x} \xrightarrow{\beta_1 \circ \beta_2} \xi_{C_R - \Gamma_x}, x_1, x_2).$$

7.2.3. *Variant 3.* Let G move. Let $\mathcal{I}$ be the Iwahori group scheme over $\mathcal{O} = k[\![t]\!]$ defined for flat $\mathcal{O}$-algebras R by

$$\mathcal{I}(R) = \text{ preimage of } B(R(t)) \text{ under } G(R) \to G(R(t)).$$

Proposition 7.4. *There exist a unique smooth affine group scheme $\mathcal{G}$ over C such that*

- $\mathcal{G}|_{C^0} = G_{C^0}$*;*
- $\mathcal{G}|_{\hat{C}_0} = \mathcal{I}$*.*

Define the *Gaitsgory's family*

$$\mathrm{Gr}_{\mathcal{G},C}(R) = \{(\xi.\beta, x) \mid x \in C(R), \xi \text{ is a } \mathcal{G}\text{-torsor on } C_R, \beta : \xi|_{C_R^\circ} \cong \xi^0|_{C_R^\circ} \text{ is a trivializtion}\}.$$

Remark. $\mathrm{Gr}_{\mathcal{G},C}|_{C^0} \cong \mathrm{Gr}_{G,C^0} \cong \mathrm{Gr}_G \times C^0$ and $\mathrm{Gr}_{\mathcal{G},C}|_0 = Fl_G$

7.3. **Nearby cycles.** Let X be a complex scheme and $f : X \to \mathbb{C}$ be an algebraic functor. Let

$$X_0 = f^{-1}(0), X^\times = f^{-1}(\mathbb{C}^\times).$$

Consider the commutative diagram

$$\begin{array}{ccccccc} X_0 & \xhookrightarrow{i} & X & \xhookleftarrow[j]{} & X^\times & \xleftarrow[\exp_X]{} & \tilde{X} \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f^\times} & & \downarrow \\ 0 & \longrightarrow & \mathbb{C} & \longleftarrow & \mathbb{C}^0 & \xleftarrow[\exp]{} & \mathbb{C}. \end{array}$$

We want some degenerate sheaves $\mathrm{Perv}(X^\times) \to \mathrm{Perv}(X_0)$. Define the *nearby cycles functor* associated to f by

$$\begin{aligned} \Psi_f : \mathrm{Perv}(X^\times) &\to \mathrm{Perv}(X_0) \\ \mathcal{F} &\mapsto i^* j_* (\exp_X)_* (\exp_X)^* \mathcal{F}[-1]. \end{aligned}$$

Theorem 7.3. *(1) For $\mathcal{F} \in D_c^b(X^\times)$, $\Psi_f(\mathcal{F}) \in D_c^b(X_0)$.*

(2) Ψ_f is perverse t-exact.

(3) Ψ commutes with proper push forwards, smooth pullback, Verdier duality and box product.

(4) If J is a smooth affine group scheme, then Ψ_f upgrades to

$$D_{J_{C^0}}(X^\times) \to D_{J_{C^0}}(X_0).$$

7.4. **Construction of Gaitsgory's functor.** Let $C = \mathbb{C}$. Gaitsgory consider the ind-scheme $\mathrm{Gr}_{G,\mathbb{C}}$ which has the property

$$\mathrm{Gr}_{G,\mathbb{C}^\times} = \mathrm{Gr}_G \times \mathbb{C}^\times, \mathrm{Gr}_{G,\mathbb{C}^0} = Fl_G.$$

Then he defines the functor

$$\begin{aligned} Z : D^b(G(\mathcal{O})\backslash G(K)/G(\mathcal{O})) &\to D^b(I\backslash G(K)/I) \\ \mathcal{F} &\mapsto \Psi(p^* \mathcal{F}[1]) \end{aligned}$$

where $p : \mathrm{Gr}_{G,\mathbb{C}^\times} \to \mathrm{Gr}_G$ is the projection and Ψ is the nearby cycle functor associated with the sturcture map $\mathrm{Gr}_{G,\mathbb{C}} \to \mathbb{C}$.

Theorem 7.4 (Gaitsgory)**.** *(1) The functor Z is monoidal (with respect to $*^I$ and $*^{G(\mathcal{O})}$) and perverse t-exact.*

(2) For $\mathcal{F} \in D^b(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$ and $\mathcal{G} \in D^b(I\backslash G(K)/I)$ there is a canonical isomorphism

$$Z(\mathcal{F}) *^I \mathcal{G} \cong \mathcal{G} *^I Z(F).$$

(3) For any $\mathcal{F} \in D^b(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$, there is a canonical isomorphism $\pi_* \circ Z(\mathcal{F}) \cong \mathcal{F}$.

(4) For any $\mathcal{F} \in D^b(G(\mathcal{O})\backslash G(K)/G(\mathcal{O}))$, the monodromy automorphism of $Z(\mathcal{F})$ is unipotent.

8. Classical Whittaker models - Elad Zelingher

8.1. **Whittaker models.** Let F be a local field or a finite field. Take $\psi : F \to \mathbb{C}^\times$ be a non-trivial continuous additive character. Define a new character

$$\psi : U_n \to \mathbb{C}^\times$$
$$\begin{pmatrix} 1 & x_1 & * & \cdots & * \\ & 1 & x_2 & \cdots & * \\ & & \ddots & \ddots & * \\ & & & \ddots & x_{n-1} \\ & & & & 1 \end{pmatrix} \mapsto \psi(x_1 + \cdots x_{n-1}).$$

Let π be an admissible representation of $\mathrm{GL}_n(F)$. A *ψ-Whittaker functional* for π is a non-zero element of $\mathrm{Hom}_{U_n}(\mathrm{Res}_{U_n} \pi, \psi)$. If π admits such functionals, π is called *generic*.

A well-known result of Gelfand-Graev, Gelfand-Kazhdan and Shalika says that if π is irreducible and generic, then its ψ-Whittaker functional is unique up to scalar. We call π with the property that π is generic and has a unique ψ-Whittaker functional a *representation of Whittaker type*. All these notions do not depend on ψ for $G = \mathrm{GL}_n$.

Proposition 8.1 (Rodier)**.** *If $\pi_1, \pi_2, \cdots, \pi_r$ are admissible representations of Whittaker type, then the parabolic induction of*

$$\pi = \pi_1 \times \cdots \times \pi_r$$

is of Whittaker type.

If π is irreducible generic, then it is of Whittaker type. By Frobenius reciprocity implies

$$\mathrm{Hom}_{U_n}(\mathrm{Res}_{U_n} \pi, \psi) \cong \mathrm{Hom}_{\mathrm{GL}_n(F)}(\pi, \mathrm{Ind}_{U_n}^{\mathrm{GL}_n(F)} \psi)$$

have dimension 1. Therefore, there exists a unique subspace denoted by $W(\pi, \psi)$ of $\mathrm{Ind}_{U_n}^{\mathrm{GL}_n(F)} \psi$ which is isomorphic to π. This space $W(\pi, \psi)$ is called the *ψ-Whittaker model* of π.

Why should one care about Whittaker models?

1. They appear in a "Fourier expansion" cusp forms. Using this one can show:
   - (a) Weak multiplicity one results: the space of cusp forms on $[\mathrm{GL}_n]$ is multiplicity free.
   - (b) Strong multiplicity one results: if π and π' are irreducible cuspidal automorphic representations such that $\pi_v \cong \pi'_v$ away from a finite set of places, the $\pi \cong \pi'$.
2. Many integral representations of L-functions rely on the Whittaker model.
3. Generic representations help one to index the L-packets that show up in the local Langlands correspondence. If G is a reductive group and Π is an L-packet of G, we require the unique generic element of Π to corresponds to the trivial character of the centralizer of the parameter of Π.

Example 8.1. (1) Irreducible supercuspidal representations of $\mathrm{GL}_n(F)$ are generic.

(2) Unramified principal series

$$\pi = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_n}$$

are generic (of Whittaker type).

8.2. **Casselman-Shalika formula.** Let $\pi = |\cdot|^{s_1} \times \cdots \times |\cdot|^{s_n}$. Recall that π is irreducible if and only if $q^{-s_i} \neq q^{-s_j - 1}$ for any i, j. Recall that π consists of functions $f : \mathrm{GL}_n(F) \to \mathbb{C}$ such that

$$f\left(\begin{pmatrix} t_1 & * & * \\ & \ddots & * \\ & & t_n \end{pmatrix} g\right) = \delta_{B_n}^{-\frac{1}{2}}\left(\begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}\right) |t_1|^{s_1} \cdots |t_n|^{s_n} f(g).$$

A *flat section* for the family of representations π is a function f such that its restriction to $\mathrm{GL}_n(\mathcal{O})$ does not depend on $s_1, \cdots, s_n$. More precisely, a flat section is a function

$$f : (s, g) \mapsto f^{(s)}(g)$$

such that

- $f^{(s)}(g) \in \pi_s$ for any $s = (s_1, \cdots, s_n)$;
- $(s, k) \mapsto f^{(s)}(k)$ does not depend on s for $k \in \mathrm{GL}_n(\mathcal{O})$.

Consider the integral

$$l_s(f^{(s)}) = \int_{U_n} f^{(s)}(w_n u)\psi^{-1}(u)du$$

where w_n is the longest element of the Weyl group. If $l_s(f^{(s)})$ converges, it defines a ψ-Whittaker functional (or the zero functional). It turns out that $l_s(f^{(s)})$ converges absolutely if s lies in a positive cone. It converges in this cone to a meromorphic function (rational functiion in $q^{-s_1}, \cdots, q^{-s_n}$). Therefore, $l_s(f^{(s)})$ has an interpretation for every s in the domain of definition of this rational function.

π_s is spherical with its unique spherical function is given by

$$f_0^{(s)}(bk) = \delta_{B_n}^{\frac{1}{2}}(b)|t_1|^{s_1} \cdots |t_n|^{s^n}.$$

It turns out that

$$\begin{aligned} l_s(f_0^{(s)}) = \prod_{1 \leq i,j \leq n} (1 - q^{-s_i + s_j - 1}) &= \det(I_{M_n(\mathbb{C})} - \mathrm{Ad} \begin{pmatrix} q^{-s_1} & & \\ & \ddots & \\ & & q^{-s_n} \end{pmatrix} q^1) \\ &= \frac{1}{L(1, \pi_s, \mathrm{Ad})}. \end{aligned}$$

Notice that it has zero exactly if $q^{-s_i+s_j-1} = 1$ for sone i and j, or equivalently when π_s is not irreducible.

Let

$$W_0^s(g) = \frac{l_s(\pi_s(g))f_0^{(s)}}{l_s(f_0^{(s)})}$$

so W_0^s is a spherical vector, i.e.,

$$W_0^s(g) = W_0^s(gk), \quad \forall g \in \mathrm{GL}_n(F), k \in \mathrm{GL}_n(\mathcal{O}).$$

We have $W_0(I_{M_n(\mathbb{C})}) = 1$.

Recall that we have the Iwasawa decomposition

$$\mathrm{GL}_n(F) = U_n A_n(\varpi) \, \mathrm{GL}_n(\mathcal{O}).$$

We then have

$$W_0^s(uak) = \psi(u) W_0^s(a).$$

So it suffices to show that

$$W_0^s(a) = W_0^s\left(\begin{pmatrix} \varpi^{m_1} & & \\ & \ddots & \\ & & \varpi^{m_n} \end{pmatrix}\right) \neq 0.$$

Because the center $F^\times$ acts by the central character, we may assume $m_1 \geq m_2 \geq \cdots \geq m_n \geq 0$, so that $m = (m_1, m_2, \cdots, m_n)$ is a partition. Let

$$\varpi^m = \begin{pmatrix} \varpi^{m_1} & & \\ & \ddots & \\ & & \varpi^{m_n} \end{pmatrix} \in A(\varpi).$$

Theorem 8.1 (Shintani, Casselman-Shalika)**.**

$$W_0^s(\varpi^m) = \delta_{B_n}^{\frac{1}{2}}(\varpi^m) \cdot S_m(q^{-s_1}, \cdots, q^{-s_n})$$

where S_m is the Schur polynomial corresponding to m.

9. Geometric Whittaker models - Pam Gu

9.1. **Categorification of Whittaker models.** Let G be a split reductive group defined over $\bar{\mathbb{F}}_p$. Let B be a Borel with unipotent radical U and $U^- \subseteq B^-$ the opposite unipotent radical in the opposite Borel. Let π be a representation of G. Fix $\psi : U^- \to \mathbb{C}$ be a nontrivial additive character.

Consider the cases when $V_\pi = \mathrm{Fun}(X, \mathbb{C})$ where X is a G-space. We categorify the Whittaker model $W(\pi, \psi)$ when π is of Whittaker type.

Step 1. Categorify ψ.

Recall that we have $\mathbb{C}/\mathbb{Z} \to \mathbb{C}^\times, z \mapsto e^{2\pi i z}$ and $\mathbb{A}^1/\mathbb{F}_p \to \mathbb{A}^1$ through the Artin-Schreier cover $a : x \mapsto x^p - x$. Moreover we have

$$a_*(\bar{\mathbb{Q}}_l) = \bigoplus_{\psi' : \mathbb{F}_p \to \mathbb{C}^\times} \mathcal{L}_\psi.$$

Here $\mathcal{L}_\psi$ is the Artin-Schreier sheaf which corresponds to ψ under the sheaf-to-function correspondence. We define a map p by the diagram

$$\begin{array}{ccccc} U^- & & \xrightarrow{\qquad\qquad p \qquad\qquad} & & \\ \downarrow & & & & \\ U^-[U^-, U^-] & \xrightarrow{\sim} & \prod_{\alpha \in \Delta} \mathbb{G}_{a,\alpha} & \xrightarrow[\Sigma]{} & \mathbb{G}_a. \end{array}$$

Let $\mathcal{L} = p^* \mathcal{L}_\psi$.

Step 2. let $U^- \curvearrowright X$ by multiplication. A $(U^-, \mathcal{L})$-equivariant complex on X is a pair $(\mathcal{F}, \beta)$ where $\mathcal{F} \in D_c^b(X)$ and

$$\beta : m^* \mathcal{F} \cong \mathcal{L} \otimes \mathcal{F}$$

is an isomorphism satisfying the usual cocycle condition.

Let $D_{(U^-.\mathcal{L})}(X)$ be the category of $(U^-, \mathcal{L})$-equivariant complexes on X. Morphisms in this cateogry are morphisms ini $D_c^b(X)$ which commute with β.

We can define averaging functors

$$D_c^b(X) \underset{\mathrm{Av}_{\mathcal{L}!}}{\overset{\mathrm{Av}_{\mathcal{L}*}}{\rightrightarrows}} D_{(U^-.\mathcal{L})}(X)$$

by

$$\mathrm{Av}_{\mathcal{L}!} := m_*(\mathcal{L} \otimes -)[\dim U^-],$$
$$\mathrm{Av}_{\mathcal{L}*} := m_!(\mathcal{L} \otimes -)[\dim U^-].$$

One can imagine these functors as smearing and integrating sheaf over the U^--orbits to make it equivariant. The forgetful functor $D_{(U^-,\mathcal{L})}(X) \to D_c^b(X)$ is fully faithful.

9.2. **Iwahori-Whittaker category, anti-spherical category and the averaging functor between them.** Let $I^- \subseteq L^+G$ be the iwahori subgroup corresponding to B^-, $I_U^- \subseteq I^-$ be the pro-unipotent radical in the unipotent radical of I^-. Let Fl be the corresponding flag variety. Let U_K and U_K^- be the unipotent radical and opposite unipotent radical of L^+G. Let $\psi : U_K^- \to \mathbb{C}$ be an additive character.

Recall that the anti-spherical module is

$$M \cong \mathcal{H} \otimes_{\mathcal{H}_{\text{fin}}} \mathbb{Z}[v, v^{-1}]$$

where $\mathcal{H} = \mathbb{Z}[v, v^{-1}][\{T_s\}, (\theta_\lambda)]$ modulo the quadratic and BZ relations and $\mathcal{H}_{\text{fin}}$ acts on $\mathbb{Z}[v, v^{-1}]$ by the sign representation. We have a realization

$$M \cong W(\text{Fun}(Fl, \mathbb{C}), \psi).$$

We expect a categorification of the realization:

$$M \cong D_{U_K^-, L}(Fl).$$

Issue. U_K^--orbits on Fl have neither finite dimension nor finite co-dimension. It is difficult for us to work with sheaves on ∞-dimensional space.

One solution is to use Drinfeld compactification (Frenkel-Gaitsgory-Vilonen). Here we use another approach, the Iwahori-Whittaker techniques.

The key idea is to replace U_K^- by I_U^-. We have for non-degenerate characters ψ of U_K^- and ψ_U of I_U^-. It follows from the isomorphism

$$W(\text{Fun}(Fl, \mathbb{C}), \mathbb{C}) \cong W_{IW}(\text{Fun}(Fl, \mathbb{C}), \psi_U).$$

that

$$M \cong W_{IW}(\text{Fun}(Fl, \mathbb{C}), \psi_n).$$

The I_U^--orbits of Fl are just opposite Bruhat cells. So passing to I_U^--orbits resolves the problem with ∞-dimension orbits.

Using same procedure as before, we have

$$D^b_{IW}(Fl) = D_{(I_U^-, \mathcal{L})}(Fl).$$

Another realization of anti-spherical module is

$$M \cong \mathcal{H}/\langle b_w \mid w \notin {}^fW \rangle$$

where b_w is the Kazhdan-Lusztig basis and ${}^fW$ is the set of minimal coset representations in $W_{\text{fin}} \backslash W$.

The cateogrification of M is given

$$M = \text{Perv}_I(Fl)/\{\text{the Serre subcateogry generated by } Fl_w^I, w \notin {}^fW\},$$

where Fl_w^I is the simple object in the image of the morphism

$$\Delta_w^I(\bar{\mathbb{Q}}_l) \to \nabla_w^I(\bar{\mathbb{Q}}_l),$$

$\Delta_w^I(\bar{\mathbb{Q}}_l)$ (resp. $\nabla_w^I(\bar{\mathbb{Q}}_l)$) is the standard (resp. co-standard) perverse sheaf on Fl.

We have an averaging functor

$$\text{Av}_{IW} : D^b_I(Fl) \to D^b_{IW}(Fl).$$

Theorem 9.1. *(1) The functor* Av_{IW} *is t-exact with respect to the perverse t-structure on* $D^b_I(Fl)$ *and* $D^b_{IW}(Fl)$.

(2) The restriction of this functor to the hearts of t-structures through a fully faithful functor

$$M \to P_{IW}(Fl)$$

where $P_{IW}(Fl)$ *is the heart of t-structure on* $D^b_{IW}(Fl)$ *and the functor is an equivalence of cateogry.*

9.3. **Iwahori-Whittaker verison of geometric Satake category.** There is a Satake equivalence

$$\mathrm{Perv}_{L^+G}(\mathrm{Gr}) \cong \mathrm{Rep}^{\mathrm{alg}}_{\bar{\mathbb{Q}}_l}(\hat{G}).$$

By Bezrukavnikov-Gaitsgory-Mirkovic-Riche-Rider, there is an equivalence of categories

$$\mathrm{Perv}_{L^+G}(\mathrm{Gr}) \cong \mathrm{Perv}_{IW}(\mathrm{Gr})$$

where $\mathrm{Perv}_{IW}(\mathrm{Gr})$ is the category of Iwahori-Whittaker perverse sheaves on Gr.