# BEZRUKAVNIKOV'S EQUIVALENCE SEMINAR - PERVERSE SHEAVES

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**Recap and Roadmap:** Let $G$ be a split reductive algebraic group, $F = \mathbb{F}_q(\!(t)\!)$, $\mathcal{O} = \mathbb{F}_q[\![t]\!]$. Let $\hat{G}$ be the Langlands dual group. The Kazhdan–Lusztig isomorphism describes the structure of the (generic) affine Hecke algebra

$$\mathcal{H} \cong K^{\hat{G}\times\mathbb{G}_m}(\widehat{\mathrm{St}}).$$

Bezrukavanikov's equivalence upgrades this to categories, but it does not (yet) have a free parameter $q$:

$$D_{\mathcal{I}}(\mathrm{Fl}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) \cong D^b\mathrm{Coh}^{\hat{G}}(\widehat{\mathrm{St}})$$

where $\widehat{\mathrm{St}}$ is the (derived) Steinberg variety for $\hat{G}$. The key step was the construction of a faithful action

$$\mathcal{H} \curvearrowright \text{anti-spherical module} \cong K^{\hat{G}\times\mathbb{G}_m}(\widetilde{\mathcal{N}})$$

where $\widetilde{\mathcal{N}}$ is the Springer resolution of the nilpotent cone for $\hat{G}$. We will categorify the latter action by constructing an equivalence of two categories:

$$D_{\mathcal{IW}}(\mathrm{Fl}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) \cong D^b\mathrm{Coh}^{\hat{G}}(\widetilde{\mathcal{N}}).$$

The left side is called the the category of Iwahori–Whittaker sheaves on $\mathrm{Fl}_{\overline{\mathbb{F}}_q}$, which after we define it will clearly be a geometrization of the anti-spherical module. The first step toward this is to categorify two descriptions of the center $Z(\mathcal{H}(G(F)//\mathcal{I})) \cong \mathcal{H}(G(F)//G(\mathcal{O}))$, via geometric Satake

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) \cong \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G}) \cong \mathrm{Coh}^{\hat{G}}(\mathrm{Spec}\,\overline{\mathbb{Q}}_\ell)$$

and Gaitsgory's central functor

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}_{\mathcal{I}}(\mathrm{Fl}_{\overline{\mathbb{F}}_q}, \overline{\mathbb{Q}}_\ell).$$

In this talk we will give background on these tools from algebraic geometry: (equivariant) perverse sheaves and derived categories.

**Derived Categories:** Let $\mathcal{A}$ be an abelian category (e.g., Mod-$R$ or $\mathrm{Sh}(X, R)$). Then $\mathcal{A}$ has a notion of *exact sequences*. An additive functor $F\colon \mathcal{A} \to \mathcal{A}'$ is *exact* if it preserves exact sequences. Many interesting functors are not quite exact. For example, the functor of global sections

$$\Gamma(X, -)\colon \mathrm{Sh}(X, R) \to \text{Mod-}R$$

is only left exact:

$$0 \to M_1 \to M_2 \to M_3 \to 0 \quad \Rightarrow \quad 0 \to \Gamma(X, M_1) \to \Gamma(X, M_2) \to \Gamma(X, M_3).$$

The derived functors of $\Gamma(X, -)$ remedy this. We have $\Gamma(X, -) = H^0(X, -)$, and there is a long exact sequence

$$0 \to H^0(X, M_1) \to H^0(X, M_2) \to H^0(X, M_3) \to H^1(X, M_1) \to H^1(X, M_2) \to H^1(X, M_3) \to H^2(X, M_1) \to \cdots$$

The $H^i(X, M)$ are defined as follows. Pick any injective resolution of $M$:

$$0 \to M \to I_0 \to I_1 \to \cdots,$$

where the complex above is acyclic and the $I_k$ are injective. Then apply $\Gamma(X, -)$ and take cohomology of the complex with $\Gamma(X, M)$ removed:

$$H^i(X, M) := \frac{\ker(\Gamma(X, I_i) \to \Gamma(X, I_{i+1}))}{\mathrm{im}(\Gamma(X, I_{i-1}) \to \Gamma(X, I_i))}.$$

This is independent of the choice of injective resolution up to canonical isomorphism.

Derived categories allow one to package the $H^i(X,-)$ into a single functor for all $i$. If $\mathcal{A}$ is a *Grothendieck abelian category*, this is constructed as follows. Let $\mathrm{Ch}^+(\mathcal{A})$ be the category of *bounded below* chain complexes in $\mathcal{A}$, meaning

$$\mathcal{C}_\bullet = \cdots \to A_{-1} \to A_0 \to A_1 \to \cdots, \quad A_i = 0 \text{ for } i \ll 0.$$

A quasi-isomorphism $\mathcal{C}_\bullet \to \mathcal{D}_\bullet$ is a map of chain complexes which induces isomorphisms on cohomology

$$H^i(\mathcal{C}_\bullet) \cong H^i(\mathcal{D}_\bullet) \in \mathcal{A}, \quad \text{for all } i.$$

The derived category $D^+(\mathcal{A})$ is the localization of $\mathrm{Ch}^+(\mathcal{A})$ at the quasi-isomorphisms, i.e. there is a functor $\mathrm{Ch}^+(\mathcal{A}) \to D^+(\mathcal{A})$ which is universal for functors sending quasi-isomorphisms to isomorphisms. The category $D^+(\mathcal{A})$ has the same objects as $\mathcal{A}$, but in general it is difficult to compute Hom spaces in a localization. Remarkably, restricting only to complex of injectives, the natural functor $\mathrm{Ch}^+(\mathrm{Inj}(\mathcal{A})) \to \mathrm{Ch}^+(\mathcal{A}) \to D^+(\mathcal{A})$ induces equivalences

$$\mathrm{Hom}_{K^+(\mathrm{Inj}(\mathcal{A}))}(\mathcal{C}_\bullet, \mathcal{D}_\bullet) = \mathrm{Hom}_{D^+(\mathcal{A})}(\mathcal{C}_\bullet, \mathcal{D}_\bullet)$$

where $K^+(\mathrm{Inj}(\mathcal{A}))$ is the *homotopy category* of $\mathrm{Inj}(\mathcal{A})$, meaning one considers two maps $\mathcal{C}_\bullet \to \mathcal{D}_\bullet$ to be equivalent if they are chain-homotopic to each other. By the definition of a quasi-isomorphism, the operation of taking the cohomology of a chain complex descends to the derived category:

$$H^i(-) \colon D^+(\mathcal{A}) \to \mathcal{A}.$$

Now given a left-exact functor $F \colon \mathcal{A} \to \mathcal{A}'$, there is a derived functor fitting into a diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{F} & \mathcal{A}' \\ \downarrow & \swarrow\!\!\!\swarrow & \downarrow \\ D^+(\mathcal{A}) & \xrightarrow{RF} & D^+(\mathcal{A}') \end{array}$$

and $\mathrm{Ch}^+(\mathcal{A}) \to D^+(\mathcal{A}) \xrightarrow{RF} D^+(\mathcal{A}')$ is initial among exact functors whose composite with $\mathrm{Ch}^+(\mathcal{A}) \to D^+(\mathcal{A})$ is equipped with a natural transformation from $\mathrm{Ch}^+\mathcal{A} \xrightarrow{F} \mathrm{Ch}^+(\mathcal{A}') \to D^+(\mathcal{A}')$. Then the $H^i(RF)$ are the usual derived functors of $F$. To compute $RF(\mathcal{C}_\bullet)$, find a bounded below chain complex of injectives $\mathcal{I}_\bullet$ with a quasi-isomorphism $\mathcal{C}_\bullet \to \mathcal{I}_\bullet$, and then set

$$RF(\mathcal{C}_\bullet) := F(\mathcal{I}_\bullet).$$

The category $D^+(\mathcal{A})$ is *triangulated*, meaning there is an auto-equivalence $[1]$ defined by $(\mathcal{C}_\bullet[1])_i = \mathcal{C}_{i+1}$, along with a notion of *exact triangles*. If $\mathcal{C}^1_\bullet \to \mathcal{C}^2_\bullet \to \mathcal{C}^3_\bullet$ is an exact triangle (for example, the image of an exact sequence in $\mathcal{A}$ under $\mathcal{A} \to D^+(\mathcal{A})$), then there is an associated long exact sequence of the cohomology groups. The derived functor $RF$ preserves exact triangles.

**Perverse Sheaves:** Let $X$ be a scheme of finite-type over a field $k$, and let $\ell \neq \mathrm{char}(k)$ be a prime. Then there is an abelian category of $\mathrm{Sh}(X) := \mathrm{Sh}(X, \overline{\mathbb{Q}}_\ell)$ of étale $\overline{\mathbb{Q}}_\ell$-sheaves on $X$. Some of the nicest sheaves are *local systems*. For coefficients in a finite ring $R$ such as $\mathbb{Z}/l^n\mathbb{Z}$, a local system on $X$ is a sheaf $\mathcal{L} \in \mathrm{Sh}(X, R)$ such that there exists an étale cover $f \colon Y \to X$ with $f^*\mathcal{L}$ isomorphic to the étale sheaf associated to a finite $R$-module. The definition of $\overline{\mathbb{Q}}_\ell$-local systems is built from this by passing to the inverse limit over $\mathbb{Z}/l^n\mathbb{Z}$, then inverting $\ell$, and finally extending scalars to $\overline{\mathbb{Q}}_\ell$ (alternatively, use the pro-étale topology). A sheaf $\mathcal{F} \in \mathrm{Sh}(X)$ is *constructible* if there exists a finite stratification $X = \sqcup_{i \in I} X_i$ by smooth, irreducible locally closed subschemes such that $\mathcal{F}\big|_{X_i}$ is a local system for all $i$.

For $f \colon X \to Y$, there are adjoint functors

$$f^* \colon \mathrm{Sh}(Y) \rightleftarrows \mathrm{Sh}(X) \colon f_*.$$

While $f^*$ is exact, $f_*$ is only left exact. In this context, the derived functor is already defined on the *bounded constructible* derived category $D^b(X) := D^b_c(X, \overline{\mathbb{Q}}_\ell)$. In general there are adjoint functors

$$f^* = Rf^* \colon D^b(Y) \rightleftarrows D^b(X) \colon Rf_*,$$

$$Rf_! \colon D^b(Y) \rightleftarrows D^b(X) \colon f^!.$$

Here $f_!$ is the functor of global sections with compact support, while $f^!$ is only defined on the derived category. There is a base change formula for $f_!$ and $f^*$, and there also a base change formula for $f_*$ and $f^!$.

When $f$ is proper we have $f_* = f_!$, and when $f$ is smooth we have $f^! = f^*[2\dim X/Y](\dim X/Y)$. There is also an internal Hom functor and derived tensor product. If $k$ is algebraically closed then the derived global sections functor satisfies $\mathrm{R}\Gamma(X,-) \cong Rf_*$ where $f\colon X \to \operatorname{Spec} k$. We have

$$R\Gamma(X,-) \cong \mathrm{RHom}_{D^b(X)}(\overline{\mathbb{Q}}_\ell,-)\colon D^b(X) \to D^b(\operatorname{Spec} k) \cong \left\{ \begin{array}{c} \text{Chain complexes of} \\ \text{finite-dimensional } \overline{\mathbb{Q}}_\ell\text{-vector spaces} \end{array} \right\}$$

The cohomology sheaves of this complex are

$$R^i\Gamma(X) = \mathrm{Hom}_{D^b(X)}(\overline{\mathbb{Q}}_\ell[i],-).$$

Let $\omega_X = f^!\overline{\mathbb{Q}}_\ell$ where $f\colon X \to \operatorname{Spec} k$. This is called a *dualizing complex*. Using this and internal Hom we define the Verdier duality functor, which is an anti-equivalence of $D^b(X)$:

$$\mathbb{D}(-) := \mathscr{R}\mathscr{H}\mathrm{om}_{D^b(X)}(-,\omega_X)$$

The functor $\mathbb{D}$ swaps $f_*$ with $f_!$ and $f^*$ with $f^!$. Note: if $X$ is smooth then $\omega_X = \overline{\mathbb{Q}}_\ell[2\dim X](\dim X)$, which leads to Poincaré duality.

If $X$ is smooth, then for any local system $\mathcal{L}$ the shifted complex $\mathcal{L}[\dim X] \in D^b(X)$ is an example of a *perverse sheaf*. In general, perverse sheaves form an abelian category

$$\mathrm{Perv}(X) \subset D^b(X).$$

Here a sequence of perverse sheaves is exact if and only if it is an exact triangle in the derived category. Very similarly to the cohomology functors $H^i(-)$, there are functors

$${}^pH^i(-)\colon D^b(X) \to \mathrm{Perv}(X)$$

which send exact triangles to long exact sequences. In particular, it makes sense to say that an object $\mathcal{F} \in D^b(X)$ lies in perverse degrees $\leq n$ or $\geq n$. Now for the definition. Let

$${}^pD^b(X)^{\leq 0} \subset D^b(X)$$

be the subcategory consisting of $\mathcal{F}$ such that there exists a stratification $X = \sqcup_{i\in I} X_i$ with the property that

$$H^n(\mathcal{F}\big|_{X_i}) \begin{cases} \text{vanishes} & n > -\dim X_i \\ \text{is a local system} & n \leq -\dim X_i. \end{cases}$$

We set

$${}^pD^b(X)^{\geq 0} = \mathbb{D}(D^b(X)^{\leq 0}), \quad \mathrm{Perv}(X) = {}^pD^b(X)^{\leq 0} \cap {}^pD^b(X)^{\geq 0}.$$

One advantage of perverse sheaves over constructible sheaves is that every perverse sheaf has finite length (just like the category of representations of an algebraic group!)

Suppose $j\colon X \to Y$ is a locally closed immersion. The *intermediate extension* functor is

$$j_{!*} := \mathrm{Im}({}^pH^0(Rj_!(-)) \to {}^pH^0(Rj_*(-)))\colon \mathrm{Perv}(X) \to \mathrm{Perv}(Y).$$

If $\mathcal{F} \in \mathrm{Perv}(X)$ is irreducible, then $j_{!*}(\mathcal{F})$ is irreducible. Note that if $X$ is smooth and irreducible and $\mathcal{L}$ is an irreducible local local system on $X$, then $\mathcal{L}\dim[X]$ is an irreducible perverse sheaf. All irreducible perverse sheaves on $Y$ are of the form $j_{!*}(\mathcal{L}[\dim X])$. If $j$ is a closed immersion, then $j_{!*} = Rj_* = Rj_!$. If $j$ is an open immersion, let $i\colon Y \setminus X \to X$ be the complement. Then $j_{!*}(\mathcal{F})$ is characterized by

$$j^*j_{!*}(\mathcal{F}) \cong \mathcal{F}, \quad i^*j_{!*}(\mathcal{F}) \in {}^pD(Y\setminus X)^{\leq -1}, \quad i^!j_{!*}(\mathcal{F}) \in {}^pD(Y\setminus X)^{\geq 1}.$$

**Equivariance:** We will exclusively be concerned with equivariant $\ell$-adic sheaves. The general setup is as follows. Let $k$ be an algebraically closed field and let $G$ be a smooth affine group scheme of finite type over $k$. Let $X$ be a reduced $k$-scheme of finite type with a $G$-action, such that there are *finitely many orbits*. We further suppose that the stabilizers for this action are *connected* subgroups of $G$. For example, $G \subset H$ could be a Borel subgroup of a reductive group $H$, and $X = H/G$ could be a flag variety for $H$.

Let $p\colon G \times X \to X$ be the projection and let $\pi\colon G \times X \to X$ be the action map. A perverse sheaf $\mathcal{F} \in \mathrm{Perv}(X)$ is *$G$-equivariant* if there exists an isomorphism $p^*\mathcal{F} \cong \pi^*\mathcal{F}$. Under our assumptions, it turns out that if one such isomorphism exists, then there is a unique isomorphism satisfying certain cocycle conditions, as in the definition of equivariant coherent sheaves. For this reason, it is appropriate to define

$$\mathrm{Perv}_G(X) \subset \mathrm{Perv}(X)$$

to be the full subcategory consisting of $G$-equivariant objects. This category is stable under taking subquotients inside $\mathrm{Perv}(X)$.

**Theorem.** The irreducible objects of $\mathrm{Perv}_G(X)$ are in bijection with $G$-orbits on $X$. If $j\colon U \to X$ is the inclusion of an orbit, the corresponding irreducible perverse sheaf is

$$j_{!*}(\overline{\mathbb{Q}}_\ell[\dim U]).$$

Unfortunately, the definition of the equivariant (bounded, constructible) derived category $D_G(X)$ is not as easy. In particular, it is *not* the derived category of $\mathrm{Perv}_G(X)$, nor is it the derived category of the abelian category of ordinary $G$-equivariant sheaves. The forgetful functor $D_G(X) \to D^b(X)$ is also not fully faithful. However, there do exist equivariant perverse truncation functors

$$^pH^i(-)\colon D_G(X) \to \mathrm{Perv}_G(X)$$

which are jointly conservative. Thus, every object in $D_G(X)$ is built from finitely many exact triangles starting from objects in $\mathrm{Perv}_G(X)$. Moreover, there is a six-functor formalism for $G$-equivariant maps.