BEZRUKAVNIKOV'S EQUIVALENCE

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ABSTRACT. These are the notes for RTG learning seminar on Bezrukavnikov's equivalence at University of Michigan in 2024 Fall. The notes taker is the only one who is responsible for all the mistakes and typos.

Contents

1. Introduction to Hecke algebra - Elad

1.1. **Bi-invariant K-Hecke algebra of G.** Let F be a non-archimedean local field with ring of integers $\mathcal{O}_F$, maximal ideal $\mathfrak{p}_F$, uniformizer ϖ_F, residual field $\mathbb{F}_q$. Let G be a split connected reductive group. Fix a Borel subgroup $B = AN$.

Definition. The *co-characters* of A is

$$X_*(A) = \mathrm{Hom}(\mathbb{G}_m, A).$$

Fix $\kappa = G(\mathcal{O}_F)$ to be a maximal compact open subgroup. Then $A(\mathcal{O}_F) = A \cap \kappa$. Let $A(\varpi) = A/A(\mathcal{O}_F)$.

Example 1.1. Let $G = \mathrm{GL}_n$ and A be the diagonal torus. Then we have identification

$$A(\varpi) = \left\{\mathrm{diag}(\varpi^{k_1}, \cdots, \varpi^{k_n}) \mid k_1, \cdots, k_n \in \mathbb{Z}\right\}.$$

We have an isomorphism $X_*(A) \cong A(\varpi)$ given by $\check{\mu} \mapsto \check{\mu}(\varpi)$.

We will talk about the algebra $\mathcal{H}(G /\!\!/ K) = \mathcal{H}(K\backslash G/K)$for some compact open subgroup $K \subseteq G(F)$, where

$$\mathcal{H}(G /\!\!/ K) = \{f \in C_c^\infty(G) \mid f(k_1 g k_2) = f(g), \forall k_1, k_2 \in K, g \in G\}.$$

Today we will be interested in the cases

- $K = G(\mathcal{O})$;
- $K =$ the Iwahori subgroup.

Why study this? We want to study representations π of $G(F)$ that have a fixed K-vector. Every such π admits a representation of $\mathcal{H}(G /\!\!/ K)$.

1.2. **Spherical Hecke algebra.** This is the case $K = G(\mathcal{O}_F)$. $\mathcal{H}(G /\!\!/ F)$ is an algebra, the product being convolution

$$(f_1 * f_2)(g) = \int_{G(F)} f_1(x^{-1}) f_2(xg) dx.$$

For $K = G(\mathcal{O}_F)$, we have the Cartan decomposition

$$G(F) = \sqcup_{\check{\mu} \in X_*(A)_{\mathrm{dom}}} G(\mathcal{O}_F) \varpi_F^{\check{\mu}} G(\mathcal{O}_F).$$

Here, $X_*(A)_{\mathrm{dom}}$ consists of *dominant* co-characters. By applying the Gelfand's trick to involution, we have that

Proposition 1.1. *$\mathcal{H}(G /\!\!/ K)$ is commutative if $K = G(\mathcal{O}_F)$.*

Example 1.2. For $G = \mathrm{GL}_n$, the involution $g \mapsto {}^t g$ fixes the diagonal matrices and sends $\mathrm{GL}(\mathcal{O}_F)$ to itself.

Remark. In general, $\mathcal{H}(G /\!\!/ K)$ is not commutative. However, if $\check{\mu}_1, \check{\mu}_2$ are dominant characters, then we have

$$\chi_{K\varpi^{\check{\mu}_1}\varpi^{\check{\mu}_2}K} = \chi_{K\varpi^{\check{\mu}_1}K} * \chi_{K\varpi^{\check{\mu}_2}K}.$$

Therefore, R_K^+, the subalgebra of $\mathcal{H}(G /\!\!/ K)$ generated by $\chi_{K\varpi^{\check{\mu}}K}$, then R_K^+ is commutative and we have

$$\mathcal{H}(G /\!\!/ K) = \mathcal{H}(\kappa /\!\!/ K) * R_K^+ * \mathcal{H}(\kappa /\!\!/ K).$$

Here $\mathcal{H}(\kappa /\!\!/ K)$ is non-commutative but finite dimensional, and R_K^+ is commutative but infinite dimensional.

1.3. **Iwahori-Matsumoto Hecke algebra.** Consider the quotient map $G(\mathcal{O}) \twoheadrightarrow G(\mathbb{F}_q)$. Let I be the pre-image of $B(\mathbb{F}_q)$ under this quotient map.

Let W be the Weyl group of G and $\tilde{W} = W \ltimes A(\varpi)$ be the extended affine Weyl group. We have the *Bruhat-Iwahori decomposition*:

$$G(F) = \bigsqcup_{w \in \tilde{W}} IwI.$$

For $w \in \tilde{W}$, $f_w = \chi_{IwI}$ form a basis of $\mathcal{H}(G(F) /\!\!/ I)$. The algebra $\mathcal{H}(G(F) /\!\!/ I)$ is isomorphic to the *Iwahori-Matsumoto algebra*. The latter is defined as follows. $\tilde{W}$ has a decomposition

$$\tilde{W} = W_{\mathrm{aff}} \rtimes \pi_1(G).$$

W_{aff} is a Coxeter group, so it is equipped with a standard presentation with generators and relations. The Iwahori-Matsumoto algebra is the Hecke algebra associated to the Coxeter system (i.e. for each simple reflection s there is a T_s, and for each generator h of $\pi_1(G)$ there is T_h, satisfying all the relations of the Coxeter system and semi-direct product except that $s^2 = 1$ is replaced by $T_s^2 = qT_s + q$). The isomorphism is given by $f_w = h \ltimes (\prod_j s_{i_j}) \mapsto T_h \cdot T_{s_{i_1}} \cdot \cdots \cdot T_{s_{i_r}}$.

Let $R_I^+ = \{f_{\varpi^{\check{\mu}}} \mid \check{\mu} \text{ is dominant}\}$. One can show f_w is always invertible. Define

$$R_I = \langle f_{\varpi^{\check{\mu}}}, f_{\varpi^{\check{\mu}}}^{-1} \mid \check{\mu} \text{ is dominant} \rangle.$$

This is a commutative subalgebra of $\mathcal{H}(G /\!\!/ I)$. Tempting to define a map

$$\begin{aligned} A(\varpi) &\to R_I^\times \\ \varpi^\mu &\mapsto f_{\varpi^\mu}. \end{aligned}$$

But this will not be a homomorphism of groups. If $\check{\mu}_1, \check{\mu}_2$ are dominant, then

$$f_{\varpi^{\mu_1}\varpi^{\mu_2}} = f_{\varpi^{\mu_1}} * f_{\varpi^{\mu_2}}.$$

Any co-character $\check{\lambda}$ is of the form $\check{\lambda} = \check{\mu}_1(\check{\mu}_2)^{-1}$ where $\check{\mu}_1$ and $\check{\mu}_2$ are dominant. For $w \in \tilde{W}$, write

$$T_w = q^{-\frac{l(w)}{2}} f_w$$

where $l : \tilde{W} \to W_{\text{aff}} \to \mathbb{Z}_{\geq 0}$ is the length. Define for $\check{\lambda}$

$$\theta(\check{\lambda}) = (T_{\check{\mu}_1}) \cdot (T_{\check{\mu}_2})^{-1}.$$

Then θ is well-defined and is a homomorphism of groups

$$\theta : A(\varpi) \to R_I^{\times}.$$

1.4. **Bernstein-Zelevinsky relation.**

Theorem 1.1. *Let s be a simple reflection in W. We have the following identity*

$$\theta(\check{\lambda})T_s - T_s\theta(s(\check{\lambda})) = T_s\theta(\check{\lambda}) - \theta(s(\check{\lambda}))T_s = (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\frac{\theta(\check{\lambda}) - \theta(s(\check{\lambda}))}{1 - \theta(\check{\alpha})^{-1}}.$$

where $\check{\alpha}$ is a certain co-character $s(\check{\alpha}) = \check{\alpha}^{-1}$.

Corollary 1.1. *The center of $\mathcal{H}(G \mathbin{/\!\!/} I)$ is*

$$\left\{ \sum_{\check{\lambda}} a_{\check{\lambda}}\theta(\check{\lambda}) \mid a_{w\check{\lambda}} = a_{\check{\lambda}}, \forall w \in W, \check{\lambda} \in X_*(A)_{\text{dom}} \right\}.$$

2. The nilpotent cone, Springer fibers & resolution and Steinberg variety

2.1. **The nilpotent cone.**

- G = complex (semisimple) reductive group, actually we are thinking about $\hat{G}(\mathbb{C})$;
- $\mathfrak{g} = \text{Lie}(G)$

Let $q : \mathfrak{g} \to \mathfrak{g}/G = \mathfrak{h}/W$ be the adjoint quotient map. The *nilpotent cone* is $\mathcal{N} = \bigcup_{0 \in \bar{\mathcal{O}}} \mathcal{O}$

Example 2.1. For $G = \text{GL}_n$, $\mathcal{N} = \{x \in \mathfrak{gl}_n \mid x^n = 0\}$. In this case, one can parametrize the nilpotent elements by

$$\{\text{nilpotent matrices}\}/\text{conjugate} \leftrightarrow \{\text{Jordan normal forms}\} \leftrightarrow \{\text{partitions of } n\}.$$

The same also works for SL_n. Take $n = 2$. There are two conjugacy classes of nilpotent matrices:

$$\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right) \rightsquigarrow \mathcal{O}_{\lambda_0} = \{\left(\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right)\}$$
$$\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right) \rightsquigarrow \mathcal{O}_{\lambda_1} = \{x \in \mathfrak{sl}_2 \mid \text{rank}(x) = 1\}.$$

Example 2.2. Let $G = \text{Sp}_{2n}$. The nilpotent orbits of Sp_{2n} can also be parametrized by certain partitions:

$\{$nilpotent orbits of $\text{Sp}_{2n}(\mathbb{C})\} \leftrightarrow \{$partition of $2n \mid$ odd partitions occur with even multiplicities$\}$.

Take $n = 4$, then we have the closure including ordering and dimensions

$$\begin{array}{cc} \mathcal{O}_{[4]} & 8 \\ | & \\ \mathcal{O}_{[2^2]} & 6 \\ | & \\ \mathcal{O}_{[2\ 1^2]} & 4 \\ | & \\ \mathcal{O}_{[1^4]} & 0 \end{array}$$

In general, for $\text{Sp}_{2n}(\mathbb{C})$, let $r_i = |\{j \in \lambda_j = i\}|$ and $s_i = |\{j \mid \lambda_j \geq i\}|$. Then

$$\dim \mathcal{O}_\lambda = 2n^2 + n - \frac{1}{2}\sum s_i^2 - \frac{1}{2}\sum_{\text{odd } i} r_i.$$

We have $\mathcal{O}_{\lambda_1} \leq \overline{\mathcal{O}_{\lambda_2}}$ if and only if $\lambda_1 \leq \lambda_2$ under dominance ordering.

Proposition 2.1. *(1) $\mathcal{N}$ is irreducible, reduced and normal;*
(2) G acts on $\mathcal{N}$ by conjugation, with finitely many orbits, each of which has even dimension.

2.2. **Springer resolution.**

Definition. The *Springer resolution* is the projection to the first factor

$$\pi : \tilde{N} = \{(x, B) \in \mathcal{N} \times \mathcal{B} \mid x \in \mathfrak{b}\} \to \mathcal{N}$$

where $\mathcal{B}$ is the variety of Borel subalgebras of $\mathfrak{g}$.

The *Springer fiber* of $x \in \mathcal{N}$ is $\pi_s^{-1}(X)$. For $x \in \mathcal{O}_\lambda$, set $F_\lambda = \pi_s^{-1}(x)$.

Example 2.3. Let $G = \mathrm{SL}_2(\mathbb{C})$. For $x \neq 0$, $\pi_s^{-1}(X) = \{*\}$. For $x = 0$, $\pi_s^{-1}(0) = \mathcal{B} = \mathbb{P}^1(\mathbb{C})$.

Proposition 2.2. $\dim F_\lambda = \frac{1}{2} \operatorname{codim}(\mathcal{O}_\lambda \subseteq \mathcal{N})$.

Proposition 2.3. $\tilde{N} \cong T^*\mathcal{B} = \{(\mathfrak{b}, v) \in B \times \mathfrak{g}^* \mid v \in \mathfrak{b}^\perp\}$

Proof. Let $\kappa : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ be the killing form. By Cartan's 2nd criterion, κ is non-degenerate. Fix a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{b}$, consider $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Since $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is non-degenerate, $\kappa|_{\mathfrak{n}_-\times\mathfrak{n}_+}$ is also a non-degenerate pairing. $\mathfrak{b}^\perp \subseteq \mathfrak{g}^*$ corresponds to $\mathfrak{n}_+$ and hence

$$\begin{aligned} T^*\mathcal{B} &= \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathfrak{g} \mid x \text{ is nilpotent}\} \\ &= \{(\mathfrak{b}, x) \in \mathcal{B} \times \mathcal{N} \mid x \in \mathfrak{b}\} \\ &= \tilde{N}. \end{aligned}$$

□

2.3. **Steinberg variety.** Let $X = \bigcup_{\lambda\in\Lambda} X_\lambda$ be stratified variety (for example $\mathcal{N} = \bigcup_{\lambda\in\Lambda} \mathcal{O}_\lambda$).

Definition. The *conormal space* is $T^*_\Lambda X = \bigcup_{\lambda\in\Lambda} T^*_\lambda X \subseteq T^*X$ where $T^*_\lambda X = \{(x, \xi) \in X_\lambda \times T^*_x X \mid \xi \text{ vanishes on } TX_\lambda\}$

Example 2.4. Consider $\mathbb{C} = \mathbb{C}^\times \cup \{0\}$. Then

$$\begin{aligned} T^*_0 &= \mathbb{C} = \{(0, y) \in \mathbb{C}^2\} \\ T^*_x &= \{(x, 0) \in \mathbb{C}^2\} \text{ for } x \neq 0. \end{aligned}$$

Hence $T^*_\Lambda\mathbb{C} = \{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$.

Proposition 2.4. *(1) $T^*_\Lambda X$ is a closed subvariety of X;*
(2) $\dim T^*_\lambda X = \dim X_\lambda + \operatorname{codim}(X_\lambda \subseteq X) = \dim X$;
(3) Irreducible components of $T^*_\Lambda X$ are in bijection with Λ.

Remark. Intersection pattern of $\overline{T^*_\Lambda X}$ is hard.

Definition. Let H be a group which acts on varieties X and Y on the left and right respectively. The *balanced product* is

$$X \times_H Y = X \times Y/((xh, y) \sim (x, hy)).$$

Remark. $X \times_H Y$ is not always a variety, but for our cases of interest it will be.

Fix a Borel $B \subseteq G$, so $\mathcal{B} = G/B$. Consider

$$\begin{aligned} G \times_B G/B &\cong G/B \times G/B \\ (g, g'B) &\mapsto (gg'B, g'B). \end{aligned}$$

The set of G-orbits on $G \times_B G/B$ is equal to the set of B-orbits on G/B. Therefore, the set of B-orbits are parametrized by the Weyl group W by the Bruhat decomposition, i.e.,

$$\mathcal{B} \times \mathcal{B} = \bigsqcup_{x\in W} \mathcal{O}_x.$$

Definition. The *Steinberg variety* is

$$\mathrm{St} = \{(\mathfrak{b}, \mathfrak{b}', x) \in \mathcal{B} \times \mathcal{B} \times \mathcal{N} \mid x \in \mathfrak{b} \cap \mathfrak{b}'\}.$$

In other words, St is the fiber product

$$\begin{array}{ccc} \mathrm{St} & \longrightarrow & \tilde{\mathcal{N}} \\ \downarrow & & \downarrow \\ \tilde{\mathcal{N}} & \longrightarrow & \mathcal{N} \end{array}.$$

Now by Proposition 2.3, we have

$$\mathrm{St} \subseteq T^*\mathcal{B} \times T^*\mathcal{B} \cong T^*(\mathcal{B} \times \mathcal{B})$$
$$((x_1, \mathfrak{b}_1), (x_2, \mathfrak{b}_2)) \mapsto (x_1, \mathfrak{b}_1, -x_2, \mathfrak{b}_2).$$

Proposition 2.5. $\mathrm{St} = \bigcup_{(\mathfrak{b}_1, \mathfrak{b}_2)} T^*_{\mathcal{O}_x, (\mathfrak{b}_1, \mathfrak{b}_2)}(\mathcal{B} \times \mathcal{B}) = \bigcup_{(\mathfrak{b}_1, \mathfrak{b}_2)} (T_{(\mathfrak{b}_1, \mathfrak{b}_2)} \mathcal{O}_x)^\perp$.

Corollary 2.1. $\mathrm{St} = \bigsqcup_{w \in W} T^*_{\mathcal{O}_w}(\mathcal{B} \times \mathcal{B})$ *is a conormal space.*