BEZRUKAVNIKOV'S EQUIVALENCE SEMINAR INTRODUCTION

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Disclaimer: This is an overview of what's to come. You don't have to understand any of it right now!

Local Langlands: Let G be a split reductive group over a non-archimedean local field F and let $\hat{G}$ be the Langlands dual group. Let W_F be the Weil group of F. The local Langlands correspondence posits a map

$$\left\{\begin{array}{c}\text{Smooth irreducible}\\ \text{representations of } G(F)\\ \text{on } \mathbb{C}\text{-vector spaces}\end{array}\right\} \xrightarrow{\text{Finite:1}} \left\{\begin{array}{c}(\rho\colon W_F \to \hat{G}(\mathbb{C}),\, e \in \operatorname{Lie}\hat{G}(\mathbb{C})),\\ \rho(\text{Frob}) \text{ is semisimple},\\ \rho(x)e\rho(x)^{-1} = |x|e,\ \forall x \in W_F\end{array}\right\}$$

Note that e must be nilpotent.

Unramified story: Let $\mathcal{O} \subset F$ be the ring of integers.

$$\begin{array}{ccc}
\left\{\begin{array}{c}\text{Representations admitting}\\ \text{a vector fixed by } G(\mathcal{O})\end{array}\right\} & \xrightarrow{1:1} & \left\{\begin{array}{c}\rho\colon W_F \twoheadrightarrow \mathbf{Z} \to \hat{G}(\mathbb{C}),\\ e = 0\end{array}\right\}\\
\sim\downarrow G(\mathcal{O})\text{-invariants} & & \sim\downarrow\\
\left\{\begin{array}{c}\text{Characters of}\\ C_c(G(\mathcal{O}\backslash G(F)/G(\mathcal{O}), \mathbb{C})\end{array}\right\} & \xrightarrow{1:1} & \left\{\begin{array}{c}\text{semisimple conjugacy}\\ \text{classes in } \hat{G}(\mathbb{C})\end{array}\right\}\\
\sim\downarrow \text{Satake} & & \\
\left\{\begin{array}{c}\text{Characters of}\\ K(\operatorname{Rep}_{\mathbb{C}}(\hat{G}))\end{array}\right\} & &
\end{array}$$

(Complexified) Grothendieck ring K of a monoidal abelian category $(\mathcal{A}, \otimes)$:

- Elements are formal sums $\sum_{n_i \in \mathbb{C}} n_i[X_i]$ where $[X_i] \in \operatorname{Iso}(\mathcal{A})$.
- If $0 \to X_1 \to X_2 \to X_3 \to 0$ is exact then $[X_2] = [X_1] + [X_3]$.
- $[X] \cdot [Y] = [X \otimes Y]$.
- Examples: $\dim\colon K(\text{Vect}_{\text{f.d.}}) \cong \mathbb{C}$, $K(\operatorname{Rep}_{\mathbb{C}}(\mathbb{G}_m)) \cong \mathbb{C}[x^{\pm 1}]$.

Categorification: $F = \overline{\mathbb{F}}_q((t))$, $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$.

$$\begin{array}{ccc}
C_c(G(\mathcal{O}\backslash G(F)/G(\mathcal{O}), \overline{\mathbb{Q}}_\ell) & \xrightarrow{\text{Satake}} & K(\operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G}))\\
K \uparrow \text{(up to normalization)} & & K \uparrow\\
\operatorname{Shv}_{L^+G}(\operatorname{Gr}, \overline{\mathbb{Q}}_\ell) & \xrightarrow[\sim]{?} & \operatorname{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})
\end{array}$$

Here Gr "$=$" $G(F)/G(\mathcal{O})$ and L^+G "$=$" $G(\mathcal{O})$ are objects in algebraic geometry. Geometric Satake says this is true if Shv means *perverse* sheaves. This equivalence is fundamental in geometric approaches to the Langlands program.

Tamely ramified with unipotent monodromy story: Let $B \subset G$ be a Borel, e.g. $B = \begin{pmatrix} * & * & * \\ & * & * \\ & & * \end{pmatrix}$.

Let $\mathcal{N} \subset \text{Lie } \hat{G}(\mathbb{C})$ be the nilpotent cone. Define the Iwahori subgroup as follows.

$$\begin{array}{ccc} \text{Iwahori } \mathcal{I} & \hookrightarrow & G(\mathcal{O}) \\ \downarrow & & \downarrow{\scriptstyle t\mapsto 0} \\ B & \hookrightarrow & G \end{array}$$

Then we have a correspondence as follows (Deligne–Langlands conjecture).

$$\begin{array}{ccc} \left\{ \begin{array}{c} \text{Representations admitting} \\ \text{a vector fixed by } \mathcal{I} \end{array} \right\} & \xrightarrow{\text{Finite:1}} & \left\{ \begin{array}{c} \rho\colon W_F \twoheadrightarrow \mathbf{Z} \to \hat{G}(\mathbb{C}), \\ e \text{ arbitrary} \end{array} \right\} \\ \sim\downarrow \mathcal{I}\text{-invariants} & & \sim\downarrow \\ \left\{ \begin{array}{c} \text{Irreducible modules for} \\ C_c(\mathcal{I}\backslash G(F)/\mathcal{I}, \mathbb{C}) \end{array} \right\} & \xrightarrow{\text{Finite:1}} & \left\{ \begin{array}{c} s \in \hat{G}(\mathbb{C}) \text{ semisimple} \\ e \in \mathcal{N}, ses^{-1} = qe \end{array} \right\}. \end{array}$$

Kazhdan–Lusztig theory: Prove Deligne–Langlands by (almost) writing $C_c(\mathcal{I}\backslash G(F)/\mathcal{I}, \mathbb{C}) = K(?)$. Two key observations:

- There exists an affine Hecke algebra $\mathcal{H}$ over $\mathbb{C}[v^{\pm 1}]$ such that $\mathcal{H}/\langle v - q^{-1/2}\rangle \cong C_c(\mathcal{I}\backslash G(F)/\mathcal{I}, \mathbb{C})$.
- $\text{Rep}_{\mathbb{C}}(\hat{G}) \cong \text{Coh}^{\hat{G}}(\text{Spec}(\mathbb{C}))$.

Then Kazhdan–Lusztig prove

$$\mathcal{H} \cong K(\text{Coh}^{\hat{G}\times\mathbb{G}_m}(\text{St})) =: K^{\hat{G}\times\mathbb{G}_m}(\text{St}).$$

Here St is the *Steinberg* variety. This is related to $\mathcal{N}$ as follows.

$$\begin{array}{ccc} \widehat{\mathcal{N}} & = & T^*(\hat{G}/\hat{B}) \curvearrowleft \hat{G} \times \mathbb{G}_m \ . \\ {\scriptstyle\pi}\downarrow {\scriptstyle\text{Springer resolution}} & & \\ \mathcal{N} & & \end{array}$$

Then $\text{St} = \widehat{\mathcal{N}} \times_{\mathcal{N}} \widehat{\mathcal{N}}$, and there is a geometrically defined convolution operation on Coh(St). To see that this is roughly related to Deligne–Langlands parameters, note for $e \in \mathcal{N}$ the centralizer

$$Z_{\hat{G}\times\mathbb{G}_m}(e) = \{(g, c) \ : \ geg^{-1} = c^{-1}e\}$$

acts on the Springer fiber $\pi^{-1}(e)$

Bezrukavnikov's equivalence: $F = \overline{\mathbb{F}}_q((t))$, $\mathbb{C} \cong \overline{\mathbb{Q}}_\ell$. First guess: Let Fl " = "$G(F)/\mathcal{I}$ be the affine flag variety. Then we might hope

$$\text{Perv}_{\mathcal{I}}(\text{Fl}, \overline{\mathbb{Q}}_\ell) \cong \text{Coh}^{\hat{G}\times\mathbb{G}_m}(\text{St}).$$

This is wrong for two reasons, one is fundamental and one is technical. The fundamental reason is that $\text{Perv}_{\mathcal{I}}(\text{Fl}, \overline{\mathbb{Q}}_\ell)$ is not closed under convolution, so we must work with larger *derived* categories. The technical reason is that

$$\text{St} = \widehat{\mathcal{N}} \times_{\mathcal{N}} \widehat{\mathcal{N}} = \widehat{\mathcal{N}} \times_{\text{Lie}(\hat{G})} \widehat{\mathcal{N}},$$

but $\widehat{\mathcal{N}} \to \text{Lie}(\hat{G})$ is not flat. To make the equivalence work, we must also work with *derived* schemes:

$$D_{\mathcal{I}}(\text{Fl}, \overline{\mathbb{Q}}_\ell) \cong D^b\text{Coh}^{\hat{G}\times\mathbb{G}_m}(\widehat{\mathcal{N}} \times^L_{\text{Lie}(\hat{G})} \widehat{\mathcal{N}}).$$

Note: You will not need to know any derived algebraic geometry to follow this seminar!

Plan of the seminar: We will follow Geordie's notes, working through Kazhdan—Lusztig theory, and its categorification up through a key input in Bezrukavnikov's equivalence: the Arkhipov–Bezrukavnikov equivalence. This is the categorification of a faithful module for $\mathcal{H}$, the antispherical module, by an equivalence

$$D_{\mathcal{IW}}(\text{Fl}, \overline{\mathbb{Q}}_\ell) \cong D^b\text{Coh}^{\hat{G}}(\widehat{\mathcal{N}}).$$

The left side is the derived category of Iwahori–Whittaker sheaves on Fl, to be discussed later.