

Siegel weil & Rallis inner product formula

02/17/2025 Jraling

Setup (we use unitary dual pair in Chowli's notes, other dual pair are similar)

- F/F₀ quadratic extension of number field $A_{F_0} = A \cdot \text{id}_F$
- V Hermitian space $\dim V = m$ $H = U(m)$
- W skew Hermitian space $\dim W = 2n$ $W = X \oplus X^\vee$
- $W = \text{Res}_{F/F_0}(V \otimes W)$ symplectic space
- $\psi: F \backslash A_F \rightarrow \mathbb{C}^\times \rightsquigarrow \omega_\psi = \bigotimes_{\mathfrak{v}} \omega_{\psi_{\mathfrak{v}}}$ (local rep of $\text{mp}(W)$)

Schrodinger model: $L = \text{Res}_{F/F_0}(V \otimes X)$

$$\omega_\psi \simeq S(L(A)) = \bigotimes_{\mathfrak{v}} S(L(F_{\mathfrak{v}}))$$

$$\psi \circ \eta: F_0^\times \backslash V|_{F_0}^\times \rightarrow \mathbb{C}^\times \xrightarrow{\text{cl}^T} F \backslash F_0$$

$$F \text{ is } X_V: A_F^\times \rightarrow \mathbb{C}^\times \text{ s.t. } X_V|_{A^\times} = \eta^{\dim m}$$

$$X_W: A_F^\times \rightarrow \mathbb{C}^\times \text{ s.t. } X_W|_{A^\times} = \eta^{\dim 2n} = 1 \rightsquigarrow X_W = 1$$

$$\begin{array}{ccc} & \text{MP}(\mathbb{R}^n)(\mathbb{R}) \cong S(\mathbb{L}(\mathbb{R}^n)) & \\ \swarrow \text{XV. 3.12} & \downarrow & \\ \mathbb{U}(\mathbb{R}^n)(\mathbb{R}) \times \mathbb{U}(\mathbb{R}^n)(\mathbb{R}) & \longrightarrow & \mathbb{S}(\mathbb{R}^n)(\mathbb{R}) \end{array}$$

$\mathbb{W}_{\mathbb{R}, \mathbb{W}}$ is a top of $\mathbb{U}(\mathbb{R}^n)(\mathbb{R}) \times \mathbb{U}(\mathbb{R}^n)(\mathbb{R})$ (omit XV. 3.12 4)

Then kernel

$$\theta: \mathbb{W} \rightarrow \mathcal{A}([\mathbb{h}, \mathbb{H}])$$

$$[\mathbb{G}] = \mathbb{h}(\mathbb{F}_0)(\mathbb{h}, \mathbb{V})$$

$$\theta \mapsto \theta_{\phi}(q, \mathbb{h}) = \sum_{r \in \mathbb{L}(\mathbb{F}_0)} (\mathbb{U}_{\mathbb{W}}(q, \mathbb{h}) \phi)(r)$$

$$\pi \subset \text{Amp}([\mathbb{h}]) \quad \phi \in \pi^{\vee}$$

$$\theta: \mathbb{W}_{\mathbb{R}, \mathbb{W}} \otimes \pi^{\vee} \rightarrow \mathcal{A}([\mathbb{H}])$$

$$(\theta, \phi) \mapsto \theta_{\phi}(\phi)(\mathbb{h}) = \langle \theta_{\phi}(\cdot, \mathbb{h}), \bar{\phi} \rangle_{\mathbb{G}}$$

$$= \int_{[\mathbb{h}]} \theta_{\phi}(q, \mathbb{h}) \bar{\phi}(q) dq \quad \uparrow \text{form in phase}$$

$(\mathbb{U}_{\mathbb{W}, \mathbb{V}} |_{\pi}) \in \mathcal{A}([\mathbb{H}])$ be the image as $\phi, \bar{\phi}$ varies.

$$\left\{ \begin{array}{l} \theta: \mathbb{W}_{\mathbb{R}, \mathbb{W}} \otimes \pi^{\vee} \rightarrow (\mathbb{U}_{\mathbb{W}, \mathbb{V}} |_{\pi}) \quad \mathbb{G}(\mathbb{R}) \times \mathbb{H}(\mathbb{R}) \text{-equiv} \\ \text{similar we obtain } (\mathbb{U}_{\mathbb{W}, \mathbb{V}} |_{\mathbb{b}}) \in \mathcal{A}([\mathbb{G}]) \end{array} \right.$$

problem: $\Theta_{W,V}(z)$ zero or not

if non-zero. Series of $\Theta_{W,V}(z)$ as $H(z)$ -repres

answer: $\Theta_{W,V}(z) \neq 0 \Leftrightarrow \left\{ \begin{array}{l} \Theta_{W,V}(z^v) \neq 0 \forall v \\ L(S_{W,V}, \chi \times \chi_V) \text{ non-trivial} \\ \text{Poles in } \mathfrak{p} \text{ part} \end{array} \right.$

Siegel-weil formula.

$$\Theta_{V,W}(1) = \text{Eisenstein series on } G$$

$\varphi \in W_4, 1 \in A[CH]$ indicates both trivial rep and trivial form

$$\Theta_{W,V}(g) = \int_{[CH]} \Theta_{W,V}(g, h) dh \rightsquigarrow Z(g, W) \text{ (classical norm)}$$

Convergence issue: $\delta = \dim(\text{maximal isotropic subspace of } V)$

weil convergence range $\left\{ \begin{array}{l} \delta > 0 \quad |V| \text{ is anisotropic} \\ \text{or } m - \delta > n \end{array} \right.$

$$m = m_0 + 2\delta$$

$$V = V_0 \oplus H\delta$$

$$\begin{array}{c} | \\ V_0 \quad \text{---} \quad W \\ m_0 \quad \quad \quad 2n \end{array}$$

$$\delta < (m_0 + 2\delta) - 2n$$

$$\Leftrightarrow \delta > 2n - m_0 \quad (\text{Sturmfeder})$$

we expect:

$$\mathcal{H}_{V,W}(1) \cong \bigoplus_v \mathcal{H}_{V,W}(1)$$

minimal $\mathbb{Z}[v]$ -free
 \downarrow

Reinit with those lifts:

$$\mathcal{G}_V \in \text{Irr}(H_V) \rightsquigarrow \mathcal{H}_{W,W}(G_V) = (\mathcal{W}_{W,W} \otimes G_V^v)_{H_V}$$

universal property:

$$\text{Hom}_{\mathcal{G}_V \text{ in } H_V}(\mathcal{W}_{W,W}, \pi_{W \otimes G_V})$$

$$\cong \text{Hom}_{\mathcal{G}_V}(\mathcal{H}_{W,W}(G_V), \pi_{W \otimes G_V})$$

Structure of $\mathcal{H}_{V,W}(1) \rightsquigarrow$ depends on prime series

(omit subscripts v) F/F δ with field

$P = M \cup N \subset G = U(n)$ SIEGEL parabol $0 < X < W$
 M SURF X^V

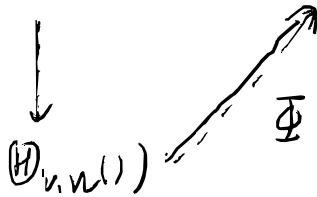
$$\begin{cases} M = \{ m(a) = \begin{pmatrix} a & & & \\ & 1 & & \\ & & \ddots & \\ & & & a^{-1} \end{pmatrix} \mid a \in \mathbb{R}_{>0} \text{ F.F.O. } (G/P) \} \\ N = \{ n(b) = \begin{pmatrix} I_n & b \\ & I_n \end{pmatrix} \mid b \in \text{Herm}_n \} \end{cases}$$

Key observation:

$$\Phi: W \cup W \longrightarrow \mathbb{I}P^G(X, S_{W \cup W}) \quad \begin{matrix} \nearrow \\ \mathbb{I}P^G(X, S_{W \cup W}) \\ \searrow \end{matrix}$$

$S_{W \cup W} = \frac{m, m+1}{2}$

$$\varphi \longmapsto \Phi_\varphi: \mathfrak{g}_1 \longrightarrow [W \cup W(\varphi) \cap \mathfrak{g}] / \mathfrak{g}$$



$\varphi \mapsto \Phi(\varphi)$
 is $H(U) \times P(X)$
 expansion

Prop: $\Phi: (D_{n,n}(1)) \longrightarrow \mathbb{I}P^G(X, S_{W \cup W})$ is injective

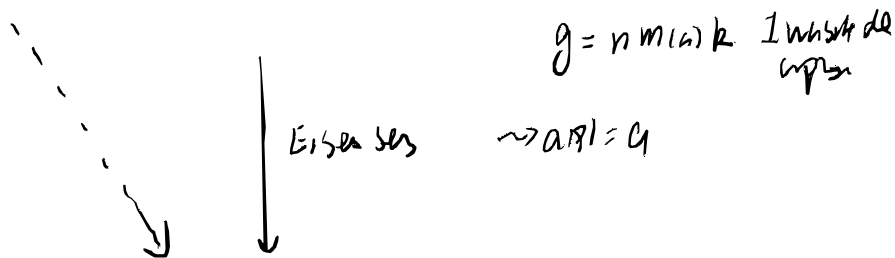
• also the strata of deepens paps see $\mathbb{I}P^G(X, S)$ is known

• str of $(D_{n,n}(1))$ is known

GHW picture: F/F_0 number field

$$\bar{\mathbb{Q}}_3: W.V.W \longrightarrow \int_{P(\mathbb{R})}^{G(A)} (S, X)$$

$$\varphi \longmapsto \mathbb{F}_{S, \varphi}: (\mathcal{P}) \longmapsto (W.V.W)(\mathcal{P}) / \text{dec}(\mathcal{P}) \quad S \text{ sum}$$



$$A[[h]]$$

$$E(g, S, \varphi) = \sum_{r \in P(\mathbb{F}_0) \setminus G(\mathbb{F}_0)} \mathbb{F}_S(r, \varphi, S)$$

• $E(g, S, \varphi)$ converges. $\Re S > 0$. meromorphic function.

• if $E(g, S, \varphi)$ is holomorphic at $S = S_0$.

$$\begin{aligned} W.V.W &\longrightarrow A[[h]] \\ \varphi &\longmapsto E(\cdot, S_0, \varphi) \end{aligned}$$

$$\begin{aligned} \text{Hom}(W.V.W, \mathbb{1} \boxtimes A[[h]]) \\ \text{GHW picture} \\ \text{is} \\ \text{Hom}_{G(\mathbb{F}_0)}(\cdot) \end{aligned}$$

Thm: (Siegel-weil)

Ass: (V, ω) is in the Weil convergence range

Then $E(g, s, \varphi)$ is holomorphic at $S_{W.V}$ and

$$E(g, S_{W.V}, \varphi) = k \theta_0(1)(g)$$

where $k = \frac{1}{2}$ if $m > n$
 $k = 1$ otherwise

Remark: Weil-Ichino-Tanaka convergence range

• not convergent $\left\{ \begin{array}{l} \text{replace } \theta_0(1) \\ \text{residue of } E(g, s, \varphi) \end{array} \right.$

Kudoh-Rallis, Gan-Din-Takeda

classical Siegel weil:

$$\theta = \sum_{n \in \mathbb{Z}} p_n^2 \quad \theta^2 \in M_1(\Gamma(1), x)$$

Ranks inner product formula.

Problem: $\phi \in \mathcal{L} \subseteq \text{Hom}(U, V)$ $\omega \in W \otimes W$

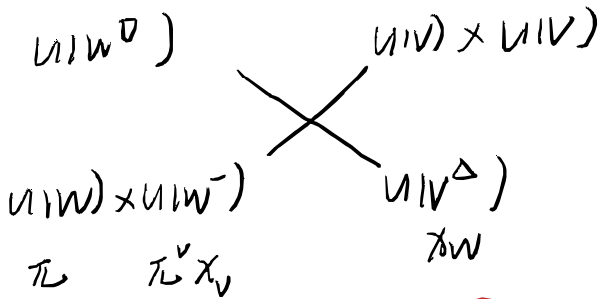
$\theta(\omega(\phi))$ non-zero?

$\theta(W \otimes V | \mathcal{L})$.

Local question: F is local field

$\mathcal{L} \subseteq \text{Hom}(U, V) = \text{Hom}(U, W)$ \rightsquigarrow $\theta(W \otimes V | \mathcal{L})$ non-zero?

see-saw diagram:



$$\begin{cases} W^{\odot} = (W, \langle \cdot, \cdot \rangle_W) \\ W^{\square} = W \otimes W^{\odot} \end{cases}$$

$$\theta(W \otimes V | \pi^{\odot} \mathcal{L}) \cong \theta(W \otimes V | \mathcal{L}) \otimes W \rightsquigarrow \theta(W \otimes V | \mathcal{L})^{\odot}$$

$$V^\Delta \otimes W^\square \cong (V \otimes W) \oplus (V \otimes W^-)$$

↙ W

$$W_{V^\Delta, W^\square} \Big|_{(U(V) \times U(W)) \times (U(V) \times U(W^-))} \cong W_{V, W} \oplus W_{V, W^-}$$

$$\text{Hom}_{(U(V) \times U(W)) \times (U(V) \times U(W^-))} (W, \pi \otimes (\pi^\vee \otimes \chi_V) \otimes \chi_W)$$

KS

$$\text{Hom}_{(U(V) \times U(W^-))} (\oplus_{V^\Delta, W^\square} (\chi_W), \pi \otimes (\pi^\vee \otimes \chi_V))$$

$$\cong \text{Hom}_{U(V)} (\oplus_{W, V} (\pi) \otimes (\oplus_{W, V} \pi^{\otimes m}) \otimes \chi_W, \chi_W)$$

$$\text{Grothm: } \oplus_{W, V} (\pi) \neq 0 \Leftrightarrow$$

$$\text{Hom}_{(U(V) \times U(W^-))} (\oplus_{V^\Delta, W^\square} (\chi_W), \pi \otimes (\pi^\vee \otimes \chi_V)) \neq 0$$

$$\text{Result: } \oplus_{V^\Delta, W^\square} (\chi_W) \longrightarrow I_p^{U(W^\square)} (\chi_V \cdot S_{V, W^\square})$$

P is the siegel parabolic of $U(W^\square)$ stabilizing

$$W^\Delta = \{ (w, -w) \mid w \in W \} \subset W^\square$$

Dunking zero in prod:

(Kronecker-Rank)

$$\text{Hom}_{U(W) \times U(W)} (I_P^{U(W^D)}, \pi_U \otimes \nu_X) \quad (*)$$

Sketch that: $U(W) \times U(W) \curvearrowright P | U(W^D)$
 \exists all the maps in W^D

$$\text{Stab}(W^D) = U(W)^\Delta$$

$$\text{open orbit: } U(W)^\Delta \backslash (U(W) \times U(W)) \xrightarrow{\cong} P | U(W^D)$$

(Grodner-Jaffe
 $M_n(F) \cong \text{GL}_n(F)$
 \downarrow
 $C_c^\infty(\text{GL}_n(F))$)

$$f \in I_P^{U(W^D)}(X_V, 0), u \in \pi, v \in \pi_V$$

$$Z(s, f, u, v) = \int_{U(W)} f(g, 1) \langle \rho(g, v) \rangle dg$$

$f \in I_P^{U(W^D)}(X_V, s)$ can represent $*$

$Z(s, \dots)$ converges $\text{Re } s > \sigma$. meromorphic function.

Kindler-Rubis Yarn

$L(s + \frac{1}{2}, \pi \cdot X_V) = \text{GLD}(Z(s, \dots))$ using good series

$$Z^*(s, f, u, v) = \frac{Z(s, f, u, v)}{L(s + \frac{1}{2}, \pi \cdot X_V)}$$

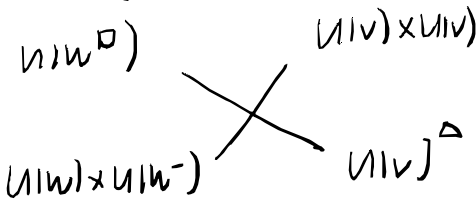
Then

$$\text{Hom}_{\mathbb{N} \times \mathbb{N} \times \mathbb{N}} \left(I_P^{(u, v, P)}(X_V, s), \pi \otimes \pi^V X_V \right) = 1$$

repn by $Z^*(s, \dots)$.

Grubhal doming see- $\leq \leq v$

(omiz λv λv from max m)



$$\begin{aligned} \varphi_1 &\in W_{W,U} \\ \bar{\varphi}_2 &\in \bar{u}|v,w \cong u|w^i \end{aligned} \quad \begin{aligned} \cdot \varphi_1 &\in \mathcal{L} \in \Delta_{\text{sup}}([h]) \\ \cdot \varphi_2 &\in \mathcal{L} \in \Delta_{\text{sup}}([h]) \rightsquigarrow \bar{\varphi}_2 \in \mathcal{L}^v \in \Delta_{\text{sup}}([h]) \end{aligned}$$

$$\langle \theta_{\varphi_1}(\varphi_1), \theta_{\varphi_2}(\varphi_2) \rangle_H$$

$$= \int_{[m]} \left(\int_{[h]} \theta_{\varphi_1}(\varphi_1, h) \bar{\varphi}_2(g, v) dg \right) \int_{[h]} \overline{\theta_{\varphi_2}(\varphi_2, h)} \varphi_2(p, v) dp dh$$

$$= \int_{[h \times h]} \underbrace{\left(\int_{[m]} \theta_{\varphi_1}(\varphi_1, h) \overline{\theta_{\varphi_2}(\varphi_2, h)} dh \right)}_{I(\varphi_1, \varphi_2)} \varphi_1(p_1) \varphi_2(p_2) dp_1 dp_2$$

$\leftarrow \theta_{v, w^0}(1)$

$$I: W_{U,W}(\bar{u}|v,w \lambda v) \rightarrow A((u|w \times u|w^-)) \subseteq A([u|w^0])$$

$$\begin{aligned} & \text{is} \\ W_{U,W} &\rightarrow I_{\text{po}}^{\text{u|w}^0}(\text{Suw}) \nearrow \varepsilon(\cdot) \quad u|v) - \text{epm} \end{aligned}$$

$$\delta: \mathcal{W}_V \times (\mathcal{W}_V) \rightarrow \mathcal{W}_V \otimes \left(\begin{array}{c} \mathcal{U}(V) \times \mathcal{U}(W) \\ \times \mathcal{U}(V) \times \mathcal{U}(W) \end{array} \right)$$

$$\phi_1 \otimes \phi_2 \quad \longmapsto \quad \delta(\phi_1 \otimes \phi_2)$$

Siegel-weil

$$I(\phi_1, \phi_2)(g_1, p_1) := E(g_1, p_1; S, W, \delta(\phi_1 \otimes \phi_2))$$

Antwort:

$$\langle \theta_{\phi_1}(\phi_1), \theta_{\phi_2}(\phi_2) \rangle$$

$$= \int_{[G \times G]} \left(\int_{[H]} \theta_{\phi_1}(p_1, h) \overline{\theta_{\phi_2}(p_2, h)} dh \right) \phi_1(p_1) \phi_2(p_2) dp_1 dp_2$$

$$= \int_{[G \times G]} \phi_1(p_1) \phi_2(p_2) E(g_1, p_1; S, W, \delta(\phi_1 \otimes \phi_2)) dp_1 dp_2$$

Erklärung

Darstellung zeta in reellen:



$$= \int_{[h \times h]} \overline{\Phi(p)} \Phi(p) \epsilon(\theta, p), \text{ sum } \delta(\phi_1, \phi_2), dp_1 dp_2$$

} unfolding

$$\int_{[h]} \langle \pi(p) \phi_1, \phi_2 \rangle_{p \text{ mod } h} \overline{\Phi}_{\text{sum}} \delta(\phi_1, \phi_2) (g, 1) dg$$

||

$$\prod_v \int_{h_v} \langle \pi(p_v) \phi_{1v}, \phi_{2v} \rangle \overline{\Phi}_{\text{sum}} \delta(\phi_{1v}, \phi_{2v}) (g_v, 1) dg_v$$

||
 $\langle \text{sum}(g) \phi_{1v}, \phi_{2v} \rangle$
 matrix coeffs of
 mod rep

↗
 $Z_v(s, \phi_{1v}, \phi_{2v}, \psi_{1v}, \psi_{2v})$

unified completion, $\langle \phi_{1v}, \phi_{2v} \rangle = 1$

ψ_{1v}, ψ_{2v} basis norm

local zeta factors

$$Z_v(s, \phi_{1v}, \phi_{2v}, \psi_{1v}, \psi_{2v}) = \frac{L(s + \frac{1}{2}, \pi_v \otimes \chi_v)}{\text{norm}(s)} \leftarrow \text{local K-fun}$$

$$Z^\#(s, \phi_{1v}, \phi_{2v}, \psi_{1v}, \psi_{2v}) = \frac{\text{norm}(s)}{L(s + \frac{1}{2}, \pi_v \otimes \chi_v)} Z(s, \phi_{1v}, \dots)$$

Thm: (Rallis-inner product)

Assume that (V, W) satisfy Weil's compat.

Condition: $\pi_i \in \text{Amp}(\Gamma_i)$ $\phi_i = \otimes_{v \in V} \phi_{i,v} \in \pi_i \in \text{Amp}^{\Gamma_i} = 1, 2$

$\phi_i = \otimes \phi_{i,v} \in S(\mathbb{C}(V))$

$$\langle \theta_{\phi_1}(\phi_1), \theta_{\phi_2}(\phi_2) \rangle_H$$

$$= \frac{L(S_{V,W} + \frac{1}{2}, \pi \times \chi_V)}{\dim(S_{V,W})} \cdot \prod_V z_V^{\dots} (S_{V,W}, \phi_{1,v}, \phi_{2,v}, \phi_{1,v}, \phi_{2,v})$$

Convolmy: Convolution rate 1

$$\textcircled{1} w.v(z) \neq 0 \text{ iff}$$

$$\cdot \textcircled{2} w.v(z) \neq 0$$

$$\cdot L(S.v.w + \frac{1}{2} \cdot z \cdot v) \neq 0$$

Mittel:

$$\langle \partial_{p_1}(A), \partial_{q_2}(A) \rangle$$

$$= \prod_v z(s, \partial_{p_1}, \partial_{q_2}, \partial_{v_1}, \partial_{v_2})$$

Distributiv: $w.v.w \rightarrow z \otimes (z^v \cdot x)$

$$G(v) \times G(w) - \text{epm}$$

$$w.v.w \otimes w.v.w \otimes z \otimes \mathbb{R} \cdot x \rightarrow \mathbb{C}$$

$$G(v) \times G(w) - \text{epm}$$

Hilf-oder