Siegel Weil & Rallis inner product formula

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Setup (we use unitary dual pair in Chao Li's notes, other dual pair are similar)

- F/F_0 quadratic extension of number field $\mathbb{A}_{F_0} = \mathbb{A}$, $\mathbb{A}_F$
- V Hermitian space. $\dim V = m$ $H = U(V)$
- W skew Hermitian space $\dim W = 2n$ $W = X \oplus X^\vee$
- $\mathbb{W} = \mathrm{Res}_{F/F_0}(V \otimes W)$ symplectic space
- $\psi : F \backslash \mathbb{A}_F \to \mathbb{C}^\times \rightsquigarrow \omega_\psi = \bigotimes_v' \omega_{\psi_v}$ weil rep of $Mp(\mathbb{W})$

Schrodinger model $\mathbb{L} = \mathrm{Res}_{F/F_0}(V \otimes X)$

$\omega_\psi \simeq S(\mathbb{L}(\mathbb{A})) = \bigotimes_v' S(\mathbb{L}(F_v))$

- $\eta : F_0^\times \backslash \mathbb{A}_{F_0}^\times \to \mathbb{C}^\times \overset{CFT}{\longleftrightarrow} F/F_0$

Fix $\chi_V : \mathbb{A}_F^\times \to \mathbb{C}^\times$ s.t. $\chi_V|_{\mathbb{A}^\times} = \eta^{\dim m}$

$\chi_W : \mathbb{A}_F^\times \to \mathbb{C}^\times$ s.t. $\chi_W|_{\mathbb{A}^\times} = \eta^{\dim 2n} = 1 \rightsquigarrow \chi_W = 1$

$$\begin{tikzcd} & Mp(W)(\mathbb{A}) \curvearrowright S(L)(\mathbb{A}) \arrow[d] \\ U(V)(\mathbb{A}) \times U(W)(\mathbb{A}) \arrow[r] \arrow[ur, "(\chi_V, \chi_W)"] & Sp(W)(\mathbb{A}) \end{tikzcd}$$

$\omega_{V,W}$ Weil rep of $U(V)(\mathbb{A}) \times U(W)(\mathbb{A})$ (omit χ_V, χ_W, ψ)

Theta kernel

$[G] = G(F_0)\backslash G(\mathbb{A})$

$$\theta : \omega \longrightarrow \mathcal{A}([G \times H])$$

$$\varphi \longmapsto \theta_\varphi(g,h) = \sum_{r \in L(F_0)} (\omega_{V,W}(g,h)\varphi)(r)$$

$\pi \subset \mathcal{A}_{cusp}([G])$ $\quad \phi \in \pi^\vee$

$$\theta : \omega_{V,W} \otimes \pi^\vee \longrightarrow \mathcal{A}([H])$$

$$(\varphi, \phi) \longmapsto \theta_\varphi(\phi)(h) = \langle \theta_\varphi(\,\cdot\,, h), \bar{\Phi} \rangle_G$$

$$= \int_{[G]} \theta_\varphi(g,h) \overline{\Phi(g)}\, dg$$

Petersson inner product

$\Theta_{W,V}(\pi) \subseteq \mathcal{A}([H])$ be the image as φ, Φ varies.

$\theta : \omega_{V,W} \otimes \pi^\vee \to \Theta_{W,V}(\pi)$ $\quad G(\mathbb{A}) \times H(\mathbb{A})$-equiv

Similarly we consider $\Theta_{V,W}(\sigma) \subseteq \mathcal{A}([G])$

problem: $\Theta_{W,V}(\pi)$ zero or not

- if non-zero, structure of $\Theta_{W,V}(\pi)$ as $H(\mathbb{A})$-reps

answer: $\Theta_{W,V}(\pi) \neq 0 \iff \begin{cases} \Theta_{W,V}(\pi_v) \neq 0 \quad \forall v \\ L(s_{V,W}+\tfrac{1}{2}, \pi \times \chi_V) \text{ non-vanish} \end{cases}$

Rallis inner product

Siegel-Weil formula

$$\boxed{\Theta_{V,W}(1) = \text{Eisens series on } G}$$

$\varphi \in W_\psi \quad 1 \in \mathcal{A}([H])$ — indicate both trivial rep and triv functs

$\theta_\varphi(1)(g) = \int_{[H]} \theta_\varphi(g,h)\, dh \rightsquigarrow I(g,\varphi)$ (choose Haar norm)

Convergence issue: $r = \dim(\text{maximal isotropic subspace of } V)$

Weil convergence range $\begin{cases} r = 0 \quad (V \text{ is anisotropic}) \\ \text{or } m - r > n \end{cases}$

$m = m_0 + 2a$

$$\begin{tikzcd}[column sep=large]
V = V_0 \oplus H_a \arrow[d, dash] & \\
V_0 \arrow[r, dash] & W \\[-20pt]
m_0 & 2n
\end{tikzcd}$$

$a < (m_0 + 2a) - 2n$

$\Longleftrightarrow$ $a > 2n - m_0$ (stable range)

We expect:

$$\Theta_{V,W}(1) \cong \otimes_v \Theta_{V_v,W_v}(1)$$

maximal H_v-iso quot

Recall local theta lift

$$\sigma_v \in \mathrm{Irr}(H_v) \rightsquigarrow \Theta_{V_v,W_v}(\sigma_v) = (\omega_{V_v,W_v} \otimes \sigma_v^\vee)_{H_v}$$

universal property:

$$\mathrm{Hom}_{G_v \times H_v}(\omega_{V_v,W_v}, \pi_v \boxtimes \sigma_v)$$

$$\simeq \mathrm{Hom}_{G_v}(\Theta_{V_v,W_v}(\sigma_v), \pi_v)$$

Structure of $\Theta_{V_v,W_v}(1) \rightsquigarrow$ degenerate principal series

(consider subspace V) F/F_0 local field

$P = MN \subset G = U(n,n)$ Siegel parabolic $0 < X < W$

M Stab $X^\vee$

$$\left\{\begin{array}{l} M = \left\{ m(a) = \begin{pmatrix} a & \\ & {}^t\bar{a}^{-1} \end{pmatrix} \;\middle|\; a \in \mathrm{Res}_{F/F_0}(GL_n) \right\} \\ N = \left\{ n(b) = \begin{pmatrix} I_n & b \\ 0 & I_n \end{pmatrix} \;\middle|\; b \in \mathrm{Herm}_n \right\} \end{array}\right.$$

Key observation:

$$\Phi: W_{V,W} \longrightarrow I_P^G(\chi_V, s_{W,V}) \hookrightarrow I_{P'}^{G'}\left((\chi_V \circ \det)\, |\det|^{s_{W,V}}\right) \qquad s_{W,V} = \frac{m-n}{2}$$

$$\begin{tikzcd} \varphi \arrow[r, mapsto] \arrow[d] & \Phi_\varphi : g \longmapsto [\omega_{V,W}(g)\varphi](0) \\ \Theta_{V,W}(1) \arrow[ur, "\Phi"'] & \end{tikzcd}$$

$\varphi \mapsto \varphi(0)$ is $H(V) \times P(X)$ equivariant

Rallis: $\Phi: \Theta_{V,W}(1) \longrightarrow I_P^G(\chi_V, s_{W,V})$ is injective

- also the structure of degenerate principal series $I_P^G(\chi, s)$ is known
- Str of $\Theta_{V,W}(1)$ is known

Global picture: F/F_0 number field

$$\Phi_s : W_{v,w} \longrightarrow I_{P(\mathbb{A})}^{G(\mathbb{A})}(s,\chi)$$

$$\varphi \longmapsto \Phi_{s,\varphi} : (g) \longmapsto (\omega_{v,w}(g)\varphi)(0)\,|\det(a(g))|^{s-s_w}$$

$g = n\,m(a)\,k$ Iwasawa decomp.

$$\begin{tikzcd} W_{v,w} \arrow[d, dashed] \arrow[r] & I_{P(\mathbb{A})}^{G(\mathbb{A})}(s,\chi) \arrow[dl, "\text{Eisenstein series}"] \\ \mathcal{A}([G]) & \end{tikzcd}$$

$\leadsto a(g) = a$

$$E(g,s,\varphi) = \sum_{\gamma \in P(F_0)\backslash G(F_0)} \Phi_s(\gamma g, s)$$

- $E(g,s,\varphi)$ converges $\operatorname{Re} s \gg 0$. meromorphic continuation.

- if $E(g,s,\varphi)$ is holomorphic at $s = s_0$.

$$W_{v,w} \longrightarrow \mathcal{A}([G])$$
$$\varphi \longmapsto E(\cdot, s_0, \varphi)$$

$\mathrm{Hom}_{G(\mathbb{A})/H(\mathbb{A})}(W_{v,w}, 1 \boxtimes \mathcal{A}([G]))$

iso

$\mathrm{Hom}_{G(\mathbb{A})}($

Thm: (Siegel-Weil)

Assume (V, W) is in the Weil convergence range

Then $E(g, s, \varphi)$ is holomorphic at $S_{W,V}$ and

$$E(g, S_{W,V}, \varphi) = \kappa\, \theta_\varphi(1)(g)$$

where $\kappa = \frac{1}{2}$ if $m > n$
$\kappa = 1$ otherwise

Remark: . Weil-Ichino-Tamura convergence range

. not convergent $\left\{ \begin{array}{l} \text{regularise } \theta_\varphi(1) \\ \text{residue of } E(g, s, \varphi) \end{array} \right.$

Kudla-Rallis . Gan-Qin-Takeda

Classical Siegel Weil:

$$\theta = \sum_{n \in \mathbb{Z}} q^{n^2} \qquad \theta^2 \in M_1(\Gamma_1(4), \chi)$$

Rallis inner product formula.

problem: $\phi \in \pi \subseteq \mathcal{A}_{cusp}([G])$ $\varphi \in \omega_{V,W}$

$\theta(\varphi|\phi)$ non-zero?

$\Theta_{W,V}(\pi)$.

Local questions: $F/\mathbb{Q}_p$ local field

$\pi \in \mathrm{Irr}(G) = \mathrm{Irr}(U(W)) \rightsquigarrow \Theta_{W,V}(\pi)$ non-zero?

see-saw diagram:

$$\begin{tikzcd}
U(W^{\square}) \arrow[dr, dash] & U(V) \times U(V) \\
U(W) \times U(W^{-}) \arrow[ur, dash] & U(V^{\Delta})
\end{tikzcd}$$

$\pi \quad \pi^{\vee}\chi_V \qquad\qquad \chi_W$

$$\begin{cases} W^{-} = (W, -\langle\,,\rangle_W) \\ W^{\square} = W \oplus W^{-} \end{cases}$$

$\Theta_{W,V}(\pi^{\vee}\chi_V) \cong \Theta_{W,V}(\pi)^{MVW}\chi_W \rightsquigarrow \Theta_{W,V}(\pi)^{\vee}$

$$V^\triangle \otimes W^\square \cong (V\otimes W)\oplus(V\otimes W^-)$$

$\swarrow W$

$$\omega_{V^\triangle, W^\square}\big|_{U(V)\times U(W)\times U(V)\times U(W^-)} \cong \omega_{V,W}\otimes\omega_{V,W^-}$$

$$\mathrm{Hom}_{U(V^\triangle)\times U(W)\times U(W^-)}\left(\omega, \pi\otimes\pi^\vee\times\upsilon\otimes\chi_W\right)$$

VS

$$\mathrm{Hom}_{U(W)\times U(W^-)}\left(\Theta_{V^\triangle, W^\square}(\chi_W), \pi\otimes(\pi^\vee\chi_V)\right)$$

$$\cong \mathrm{Hom}_{U(V^\square)}\left(\Theta_{W,V}(\pi)\otimes(\Theta_{W,V}(\pi))^{MVW}\chi_W), \chi_W\right)$$

Gross: $\Theta_{W,V}(\pi)\neq 0 \Leftarrow$

$$\mathrm{Hom}_{U(W)\times U(W^-)}\left(\underset{\overset{\shortparallel}{1}}{\Theta_{V^\triangle, W^\square}(\chi_W)}, \pi\otimes\pi^\vee\chi_V)\right)\neq 0$$

Recall: $\Theta_{V^\triangle, W^\square}(\chi_W) \longrightarrow I_P^{U(W^\square)}(\chi_V, S_{V,W^\square})$

P is the siegel parabolic of $U(W^\square)$ stabling

$W^\triangle = \{(w,-w) \mid w\in W\} \subset W^\square$

Doubling zeta integral:

(Kudla - Rallis)

$$\mathrm{Hom}_{U(W)\times U(W^-)}\left(I_P^{U(W^\square)}(\chi_V, s),\ \pi\otimes\pi^\vee\chi_V \right) \quad (*)$$

Similar thing: $U(W)\times U(W^-) \curvearrowright P\backslash U(W^\square)$

$\ni$ all the Lagrangians in $W^\square$

$$\mathrm{Stab}_{U(W)\times U(W^-)}(W^\Delta) = U(W)^\Delta$$

open orbit: $U(W)^\Delta\backslash U(W)\times U(W^-) \hookrightarrow P\backslash U(W^\square)$

$\cong$

$U(W)$

(Godement-Jacquet

$M_n(F) \subseteq GL_n\times GL_n$

$\int$

$C_c^\infty(GL_n(F))$)

$f\in I_P^{U(W^\square)}(\chi_V, 0)$, $u\in\pi$, $v\in\pi^\vee\chi_V$

$$Z(s, f, u, v) = \int_{U(W)} \hat{f}_s(g, 1)\langle \pi(g) u, v\rangle\, dg$$

$f_s\in I_P^{U(W^\square)}(\chi_V, s)$ $\rightsquigarrow$ represents element in $*$

$Z(s,\dots)$ converges $\mathrm{Re}\, s \gg 0$ meromorphic continuation.

Kudla-Rallis Yamana

$$L(s+\tfrac{1}{2}, \pi \cdot \chi_V) = GCD\,(Z(s, \dots))$$ using good sections

$$Z^{\#}(s, f, u, v) = \frac{Z(s, f, u, v)}{L(s+\tfrac{1}{2}, \pi \cdot \chi_V)}$$

Thm

$$\mathrm{Hom}_{U(W) \times U(W)}\left(I_P^{U(W^{\square})}(\chi_V, s),\ \pi \otimes \pi^{\vee} \chi_V\right) = 1$$

spanned by $Z^{\#}(s, \dots)$.

Global doubling see-saw (omit χ_V χ_W from now on)

$$\begin{matrix} U(W^{\square}) & & U(V)\times U(V) \\ & \times & \\ U(W)\times U(W^{-}) & & U(V)^{\Delta} \end{matrix}$$

$\varphi_1 \in W_{W,V}$ $\quad \phi_1 \in \pi \in \mathcal{A}_{cusp}([G])$

$\Phi_2 \in \overline{W_{V,W}} = W_{V,W^\vee}$ $\quad \phi_2 \in \pi \in \mathcal{A}_{cusp}([G]) \rightsquigarrow \overline{\phi_2} \in \pi^\vee \in \mathcal{A}_{cusp}([G])$

$$\langle \theta_{\varphi_1}(\phi_1), \theta_{\varphi_2}(\phi_2) \rangle_H$$

$$= \int_{[H]} \left(\int_{[G]} \theta_{\varphi_1}(g_1,h)\, \Phi_1(g_1)\, dg_1 \right) \overline{\int_{[G]} \theta_{\varphi_2}(g_2,h)\, \phi_2(g_2)\, dg_2}\, dh$$

$$= \int_{[G\times G]} \Big(\underbrace{\int_{[H]} \theta_{\varphi_1}(g_1,h)\, \overline{\theta_{\varphi_2}(g_2,h)}\, dh}_{I(\phi_1\otimes\phi_2)} \Big) \Phi_1(g_1)\, \phi_2(g_2)\, dg_1\, dg_2$$

$\Theta_{V,W^{\square}}(1)$

$$I: W_{V,W} \otimes \overline{W_{V,W}} \longrightarrow \mathcal{A}([U(W)\times U(W^{-})]) \subseteq \mathcal{A}([U(W^{\square})])$$

$\cong$ $\quad i(\ )$ $\quad U(V)$-eqvt

$$W_{V,W^{\square}} \longrightarrow I_{P^{\Delta}}^{U(W^{\square})}(s_{V,W})$$

$$\delta: W_{V,n} \otimes \overline{W_{V,n}} \to \simeq W_{V,n,\square}\big|_{U(V)\times U(W)\times U(V)\times U(\bar W)}$$

$$\phi_1 \otimes \phi_2 \longmapsto \delta(\phi_1 \otimes \phi_2)$$

Siegel-Weil

$$I(\phi_1, \phi_2)(g_1, g_2) := E((g_1, g_2), S_{V,W}, \delta(\phi_1 \otimes \phi_2))$$

Control:

$$\langle \theta_{\phi_1}(\phi_1), \theta_{\phi_2}(\phi_2) \rangle$$

$$= \int_{[G\times G]} \left(\int_{[H]} \theta_{\phi_1}(g_1, h)\, \overline{\theta_{\phi_2}(g_2, h)}\, dh \right) \overline{\Phi_1(g_1)}\, \Phi_2(g_2)\, dg_1\, dg_2$$

$$= \int_{[G\times G]} \overline{\Phi_1(g_1)}\, \Phi_2(g_2)\, E((g_1, g_2), S_{V,W}, \delta(\phi_1 \otimes \phi_2))\, dg_1\, dg_2$$

GWW,
Doubling zeta integral:

$$= \int_{[G\times G]} \overline{\phi_1(g_1)}\, \phi_2(g_2)\, E\big((g_1,g_2), s, \omega; \delta(\phi_1\otimes\phi_2)\big)\, dg_1\, dg_2$$

unfolding

$$\int_{[G]} \langle \pi(g)\phi_1, \phi_2 \rangle_{\text{Pet}}\, \Phi_{s,\omega,\delta(\phi_1\otimes\phi_2)}(g,1)\, dg$$

$$\|$$

$$\prod_v \int_{G_v} \langle \pi_v(g_v)\phi_{1v}, \phi_{2v} \rangle\, \Phi_{s,\omega,\delta(\phi_{1v}\otimes\phi_{2v})}(g_v,1)\, dg$$

$$\| \qquad \langle w_{\omega}(g_v)\phi_{1v}, \phi_{2v}\rangle$$

matrix coeffs of weil rep

$$Z_v(s, \phi_{1,v}, \phi_{2,v}, \varphi_{1v}, \varphi_{2v})$$

unramified computation $\langle \phi_{1v}, \phi_{2v}\rangle = 1$

$\varphi_{1v}, \varphi_{2v}$ basis vectors

local zeta integrals

$$Z_v(s, \phi_{1v}, \phi_{2v}, \varphi_{1v}, \varphi_{2v}) = \frac{L(s+\frac{1}{2}, \pi_v\otimes\chi_v)}{b_{2n,v}(s)}$$

ratio of L-fns

$$Z_v^{\#}(s, \phi_{1v}, \phi_{2v}, \varphi_{1v}, \varphi_{2v}) = \frac{b_{2n,v}(s)}{L(s+\frac{1}{2}, \pi_v\otimes\chi_v)}\, Z(s, \phi_{1v}, \dots)$$

Thm: (Rallis-inner product)

Assume that $(V, W^{\square})$ satisfy Weil's conv. part.

Condition: $\pi_i \in \mathcal{A}_{cusp}([G])$, $\phi_i = \otimes \phi_{i,v} \in \pi_i \in \mathcal{A}_{cusp}^{[G]}$, $i = 1, 2$

$\varphi_i = \otimes \varphi_{i,v} \in S(\mathbb{L}(\mathbb{A}))$

$$K \langle \theta_{\varphi_1}(\phi_1), \theta_{\varphi_2}(\phi_2) \rangle_H$$

$$= \frac{L(s_{V,W} + \frac{1}{2}, \pi \times \chi_V)}{b_m(s_{V,W})} \cdot \prod_v Z_v^*(s_{V,W}, \phi_{1,v}, \phi_{2,v}, \varphi_{1,v}, \varphi_{2,v})$$

Corolly: Conventen rule:

$\Theta_{w,v}(\lambda) \neq 0$ iff

- $\Theta_{w,w}(\lambda^\vee) \neq 0$
- $L(S_{v,w} + \frac{1}{2}, \lambda \chi_v) \neq 0$

Main.

$$\langle \Theta_{\phi_1}(f_1), \Theta_{\phi_2}(f_2) \rangle = \prod_v Z(s, S_{\phi_1,\phi_2,v}, f_{1v}, f_{2v})$$

Distinctions: $W_{v,w} \to \lambda \otimes (\lambda^\vee \chi)$ $G(V) \times G(W)$-eqv

$W_{v,w} \otimes W_{v,w}^\vee \otimes \lambda \otimes \lambda^\vee \chi \to \mathbb{C}$ $G(V) \times G(W)$ -eqv

$H(V)$-orders