THETA DICHOTOMY

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- F_0 number field.
- F/F_0 quadratic extension.
- $0 \neq \delta \in F^{\times}$ - a trace zero element.
- W - split skew–herimitan space of dimension $2n$,
- V - non-degenerate hermitian of dimension $m = 2n$.
- $G = U(W)$
- $H = U(V)$
- $\psi_0 \colon F_0 \backslash \mathbb{A}_{F_0} \to \mathbb{C}^{\times}$ a non-trivial additive character. Define $\psi \colon F \backslash \mathbb{A}_F \to \mathbb{C}^{\times}$ by $\psi(x) = \psi_0(\mathrm{tr}(\delta x))$.

For place v, let $\eta_v \colon F_{0v}^{\times}/N(F_v^{\times}) \to \{\pm 1\}$ be the quadratic character. We define

$$\varepsilon(V_v) = \eta_v(\mathrm{disc} V_v),$$

where

$$\mathrm{disc} V_v = (-1)^{\frac{m(m-1)}{2}} \det V_v \in F_{0v}^{\times}/N\left(F_v^{\times}\right).$$

Given a sequence $(U_v)_v$ of non-degenerate hermitian spaces of dimension m, there exists a non-degenerate hermitian space $\mathbb{U}$ over F such that $\mathbb{U}_v = U_v$ if and only if the following two conditions are satisfied:

(1) $\varepsilon(U_v) = 1$ for all but finitely many v.
(2) $\prod_v \varepsilon(U_v) = 1$.

Let π be an irreducible cuspidal automorphic representation of G. When is $\Theta_V(\pi) \neq 0$?

By Jialiang's talk we need:

(1) $L\left(\frac{1}{2}, \pi\right) \neq 0$.
(2) $\prod_v Z_v^{\#}(0) \neq 0$.

Theta dichotomy: for every non-split place v there exists a unique local non-degenerate hermitian space $V_v = V_v(\pi_v)$ of rank $2n$ over F_v such that $Z_v^{\#}(0) \neq 0$. If v is a non-split finite place then we can pinpoint the space exactly using the root number. We have that $Z_v^{\#}(0) \neq 0$ if and only if

$$\varepsilon(V_v) = \omega_{\pi_v}(-1) \cdot \varepsilon\left(\frac{1}{2}, \pi_v, \psi_v\right).$$

This also holds in the archimedean case, but in the archimedean case, $\varepsilon(V_v)$ does not determine the space V_v.

For any irreducible cuspidal automorphic representation $\pi = \bigotimes_v \pi_v$ of $G(\mathbb{A}_F)$ we obtain a sequence of hermitian spaces $(\mathbb{V}(\pi_v))_v$ such that $\Theta_{\mathbb{V}(\pi_v)}(\pi_v) \neq 0$. Denote

$$\varepsilon(\mathbb{V}_\pi) \coloneqq \prod_v \varepsilon(\mathbb{V}(\pi_v)).$$

We therefore have the equality

$$\varepsilon(\mathbb{V}_\pi) = \varepsilon\left(\frac{1}{2}, \pi\right),$$

where

$$\varepsilon\left(\frac{1}{2}, \pi\right) = \prod_v \varepsilon\left(\frac{1}{2}, \pi_v, \psi_v\right) = \prod_v \omega_{\pi_v}(-1) \cdot \varepsilon\left(\frac{1}{2}, \pi_v, \psi_v\right).$$

By the remark above, we can glue $(\mathbb{V}(\pi_v))$ to a hermitian space V over F if and only if We have

$$\varepsilon(\mathbb{V}) = 1,$$

which is equivalent to

$$\varepsilon\left(\frac{1}{2},\pi\right)=1.$$

If $\varepsilon\left(\frac{1}{2},\pi\right)=1$, we say that π (or $\mathbb{V}_\pi$) is *coherent*. In this case, the global theta lift of π from $G(\mathbb{A})$ to $H(\mathbb{A})$ (where $H=U(V)$) is non-zero if and only if

$$L\left(\frac{1}{2},\pi\right)\neq 0.$$

In which case, we have that $\Theta_U(\pi)=0$ for any other non-degenerate hermitian space U over F of dimension $2n$.

If $\varepsilon\left(\frac{1}{2},\pi\right)=-1$, we say that π (or $\mathbb{V}_\pi$) is *incoherent*. In this case, the global theta lift of π from $G(\mathbb{A})$ to $H(\mathbb{A})$ is zero for any $H=U(V)$, where V is any non-degenerate hermitian space over F. One of the goals of this seminar is to show that

$$L'\left(\frac{1}{2},\pi\right)\neq 0$$

implies that the *arithmetic theta lift* is not zero.