RTG NUMBER THEORY: BEZRUKAVNIKOV'S EQUIVALENCE
FALL 2024

This seminar will run Mondays at 3:00pm in EH 3088, organized jointly by Robert Cass, Charlotte Chan, Kartik Prasanna, and Elad Zelingher. Each talk should be 60 minutes long; running over due to too much audience engagement is acceptable, but 4:15pm is a strict upper bound.

The aim of this seminar is to casually introduce some ideas in geometric representation theory, with an emphasis on applications to Langlands. The path to doing this will be to understand the content of Bezrukavnikov's equivalence.

Theorem (Bezrukavnikov [B]). *Let G be a split reductive group over an algebraically closed field of characteristic p. Let $\hat{G}$ be the Langlands dual group over $\overline{\mathbb{Q}}_\ell$, for $\ell \neq p$. Then we have an equivalence of categories*

$$D_{\mathcal{I}}(\mathrm{Fl}, \overline{\mathbb{Q}}_\ell) \cong D^b \operatorname{Coh}^{\hat{G}}(\hat{\mathcal{N}} \times^L_{\hat{\mathfrak{g}}} \hat{\mathcal{N}}).$$

The left side is the derived category of Iwahori-equivariant étale sheaves on the affine flag variety for G. The right side is the derived category of $\hat{G}$-equivariant coherent sheaves on a version of the Steinberg variety for $\hat{G}$. Starting from basic algebraic geometry, we will build our understanding of all of these terms and more throughout the seminar, roughly following the route paved by Geordie Williamson's year-long course 2019-2020 [W].

Week 0: Introduction and planning. (Robert Cass) [August 26]
Overview of the seminar.

Week 1: Local Langlands and affine Hecke algebras. (Elad Zelingher) [September 9]
Lecture 11 of [W]. Overview of the Local Langlands program and affine Hecke algebras.

Week 2: Affine Hecke algebras and Deligne–Langlands. [September 16]
Lectures 11.4 through 12.2 of [W]. Introduce the extended affine Weyl group and the Iwahori–Matsumoto Hecke algebra over $\mathbb{Z}[v^{\pm 1}]$. Give examples. Discuss the center and Bernstein's presentation. Relate the Iwahori–Matsumoto Hecke algebra to $C_c(\mathcal{I}\backslash G(F)/\mathcal{I}, \mathbb{C})$ and state the Deligne–Langlands conjecture. Additional resource: [HKP].

Week 3: Springer fibers and the Steinberg variety. [September 23]
Lectures 12.3 through 13.4 of [W]. Introduce the nilpotent cone, the Springer resolution, Springer fibers, and the Steinberg variety. Give examples.

Week 4: Kazhdan–Lusztig isomorphism. [September 30]
Lectures 15.1-15.3 (plus perhaps some part of Lecture 14) of [W]. Introduce the Grothendieck group $K^G(X)$. Explain how to get a ring structure using geometric convolution and state the Kazhdan–Lusztig isomorphism. Identify nice subalgebras inside $K^{\hat{G}\times\mathbb{G}_m}(\mathrm{St})$.

Week 5: Kazhdan–Lusztig isomorphism, continued. [October 7]

Lectures 21 and 22 of [W]. Show that $K^{\hat{G}\times\mathbb{G}_m}(\mathrm{St})$ and the Iwahori–Matsumoto Hecke algebra have bases in bijection. Define the anti-spherical module and discuss its importance for Kazhdan–Lusztig. Time permitting, discuss the example of SL_2 and/or the Demazure–Lusztig formula.

Week 6: A crash course on derived categories, perverse sheaves, and K-groups. (Robert Cass) [October 21]

Lectures 17 and 24 of [W]. Start with a recap and explain what's to come. Then discuss derived categories and (equivariant) perverse sheaves with examples. Explain how these techniques are used to categorify Hecke algebras via the function-sheaf correspondence or K-groups.

Week 7: Geometric Satake. [October 28]

Lecture 10.4 of [W] plus Chapter I of [BR]. Recall the classical Satake isomorphism and state the geometric Satake equivalence $\mathrm{Perv}_{L^+G}(\mathrm{Gr}, \overline{\mathbb{Q}}_\ell) \cong \mathrm{Rep}_{\overline{\mathbb{Q}}_\ell}(\hat{G})$. Discuss with examples how the structure of $\mathrm{Rep}(\hat{G})$ appears on the perverse sheaf side, including semisimplicity and the parametrization of irreducibles by highest weight. Time permitting, additional topics could include the construction of the equivalence via the fiber functor and Tannakian formalism, the commutativity constraint, or the constant terms and semi-infinite orbits.

Week 8: Gaitsgory's central sheaves. [November 4]

Lecture 23 of [W]. State Bernstein's isomorphism for the center of the Iwahori Hecke algebra. State the existence of the nearby cycles functor and its key properties. Apply this to Gaitsgory's Beilinson–Drinfeld family over $\mathbb{A}^1$ to get a functor $\mathrm{Perv}_{L^+G}(\mathrm{Gr}, \overline{\mathbb{Q}}_\ell) \to \mathrm{Perv}_{\mathcal{I}}(\mathrm{Fl}, \overline{\mathbb{Q}}_\ell)$. Discuss centrality and compatibility with pushforward back to Gr. Time permitting, discuss the example with $\mathbb{P}^1$ deforming to a nodal curve (we won't have time to discuss Beilinson gluing). Additional resources include [G] and Chapters 2–3 of [AR].

Week 9: Classical Whittaker models. (Elad Zelingher) [November 11]

Part of Lecture 32 of [W] and Section 6 of [HKP]. Introduce classical Whittaker models, including the Whittaker model of the antispherical module and the Casselman–Shalika formula. Time permitting, an additional topic could be the geometric Casselman–Shalika formula [NP].

Week 10: Geometric Whittaker models. [November 18]

Part of Lecture 32 of [W]. Introduce categorifications of Whittaker and Iwahori–Whittaker models. Construct the antispherical and Iwahori–Whittaker categories, and the averaging functor between them. An additional resource is Section 6.4 of [AR]. Time permitting, an additional topic could be the Iwahori–Whittaker version of geometric Satake [BGMRR].

Week 11: Coherent sheaves. [November 25]

Lecture 29 of [W]. Discuss coherent sheaves on G/B, G/U and $\overline{G/U}$. Time permitting, additional topics could include the geometry of Schubert varieties in G/B or the Borel–Weil–Bott theorem.

Week 12: Construction of the functor. (Robert Cass) [December 2]

Lectures 31-32 of [W]. Give the construction of the Arkhipov–Bezrukavnikov functor and discuss the role of Whittaker models in proving it is an equivalence.

Week 13: Summary plus epsilon. (Charlotte Chan) [December 9]

Review the semester and discuss Bezrukavnikov's equivalence.

References

[AR] P. Achar and S. Riche. *Central sheaves on affine flag varieties.* `https://lmbp.uca.fr/~riche/central.pdf`

[BR] P. Baumann and S. Riche. *Notes on the Geometric Satake Equivalence.* `https://arxiv.org/pdf/1703.07288`.

[B] Roman Bezrukavnikov. On two geometric realizations of an affine Hecke algebra. *Publ. Math. Inst. Hautes Études Sci.*, 123:1–67, 2016. `doi:10.1007/s10240-015-0077-x`.

[BGMRR] Roman Bezrukavnikov, Dennis Gaitsgory, Ivan Mirković, Simon Riche, and Laura Rider. An Iwahori-Whittaker model for the Satake category. *J. Éc. polytech. Math.*, 6:707–735, 2019. `doi:10.5802/jep.104`.

[G] D. Gaitsgory. Construction of central elements in the affine Hecke algebra via nearby cycles. *Invent. Math.*, 144(2):253–280, 2001. `doi:10.1007/s002220100122`.

[HKP] T. Haines, R. Kottwitz and A. Prasad. *Iwahori–Hecke algebras.* `https://arxiv.org/pdf/math/0309168`

[NP] B. C. Ngo and P. Polo. *Résolutions de Demazure affines et formule de Casselman-Shalika géométrique.* `https://arxiv.org/abs/math/0005022`

[W] A. Romanov and G. Williamson. *Langlands correspondence and Bezrukavnikov's equivalence.* `https://arxiv.org/pdf/2103.02329`.